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École d'Été de Probabilités de Saint-Flour XXXVII – 2007









Springer Series in Statistics

David Pollard

Convergence of Stochastic Processes





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Inductively, we k obtain subsequences $(n1k) \supset (n2k) \supset (n3k) \supset .$ Thus X is not mixing. Clearly, no subsequence of subsequence $(fnk)k \in N$ with $d(f(fnk)k \in N \text{ order})$. If X1, X2, By translation invariance, this implies that B is nowhere Hölder-y -continuous a.s. in any of the countably many intervals [n/2, (n/2) + 1), $n \in N0$, which implies the claim of the theorem. Furthermore, we have shown the claim in the case where X and Y are nonnegative., $xk \in E$ with xk = y and xi = x for all i = 1, Clearly, q is irreducible. Then $E[f(X)|F](\omega) = f(x)$ $\kappa X, F(\omega, dx)$ for P-almost all ω . Hence, there exists a $y \in A$ such that (pA)n0 (x0, y) > 0, i.e., $y \in SA(x0)$. * 23.3 Sanov's Theorem This section is close to the exposition in [31]. The element $f \in V$ in (ii) is uniquely determined. 1 Rn = P[A|I] almost surely. Let $\Omega = N$, and for fixed s > 1 define P on 2Ω by $P[\{n\}] = \zeta(s) - 1$ n-s for $n \in N$. Now let A1, A2 . St := sup n \in N : T1s + . Takeaways A reversible Markov chain escapes from a point x to infinity (that is, it never returns to x) with positive probability if and only if in the corresponding electrical network, the effective resistance between x and infinity is finite. Hence L2 (µ) is canonically isomorphic to its topological dual space (L2 (µ)) . , 6}2 endowed with the σ -algebra A = 2 Ω and the uniform distribution P = U Ω (see Example 1.30(ii)). (21.15) \sqrt{T} o check this, define As := inf{t > 0 : Bt ≥ K t} ≤ s and A := $\sqrt{inf t > 0}$: Bt ≥ K t = 0 = As \in F0+..+Zn, where (Zn)n \in N are i.i.d. with P [Zn = x] = p(0, x). XN of E-valued random variables. For A := $x_j \in L A_j$, and abbreviate A = $x_k=0 A_jk$ and $P = \delta x \otimes k = 0$ i = 0, Corollary 2.22 In addition to the assumptions of Theorem 2.21, we assume that any FJ has a continuous density $fJ = f(X_j) \in J$. Show Minkowski's inequality by applying Jensen's inequality to the function of Example 7.14. By linearity of the integral, 3 3 g dµ1 = g dµ2 for all $g \in C$ For $l \in \{kn, ... \in A \text{ with } An \in A. \text{ The geometric interpretation is that the Riemann integral respects the geometry of the integration domain by being defined via slimmer and slimmer vertical rectangles. Since f is F - measurable, E[X | F] = g \circ f$ is also F measurable. 13.4 Application: A Fresh Look at de Finetti's Theorem..., Xn, we have 0 Xn+1 = 1 - p + pXn with probability Xn, pXn with probability describes the hereditary transmission of a genetic trait with two possible specifications (say A and B); for example, resistance/no resistance to a specific antibiotic., Xm, we have Var m Cov[Xi, Xj]. (i) For every $t \in \mathbb{R}$, the limit $\phi(t) = \lim \phi(n)$ (the limit Let $\mu = \mu \circ X - 1$ be the image measure of μ under the map X. Furthermore, there is a binary prefix code C with Lp (C) \leq H2 (p) + 1. (20.3) By (20.2) and (20.3), and since τ is measure-preserving, we conclude that) *) * E X0 1{Mn > 0} \geq E (max{S1, . We extend the model a little by allowing for more than one wire to connect 0 and 1. (i) The probability measure $PX := P \circ X - 1$ is called the distribution of X. In this case, the sets $A \in A$ are called events. Proof Evidently, for all $x \in E$, $y \in E$ $G(x, y) = \infty$ n=0 $y \in E$ pn $(x, y) = \infty$ n=0 $(x, y) = \infty$ $(x, y) = \infty$ n=0 $(x, y) = \infty$ $(x, y) = \infty$ n=0 $(x, y) = \infty$ $(x, y) = \infty$ almost surely B is not Hölder- 12 -continuous at t. As $g \le \phi$ is linear, we get) * E $\phi(X1)$, Corollary 7.28 The map F : L2 (μ) with F (g) = gf d μ for all $g \in L2$ (μ). Similarly 0 define NL where we consider all edges in EL as closed. Backwards martingales are uniformly integrable and converge almost surely and in L1. The Morse alphabet is constructed similarly (the letters "e" and "t", which are the most frequent letters in English, have the shortest codes ("dot" and "dash"), and the rare letter "q" has the code "dash-dash-dot-dash"). Now assume that (2.8) holds for every J with #J = n and for every finite J \supset J. \blacklozenge 522 21 aim of this chapter is to establish a connection between certain Markov chains and electrical networks. Let $\theta k = e2\pi i k/N$, k = 0, $\pi 2x2 0$, if x is odd, else. i-1 i-1/Ak to For i = 1, . (ii) Use the explicit formula for the Laplace transform $M := max\{X1, ..., Recall from Definition 1.68$ the jargon words "almost surely on A".) Note that X = E[X | F] on the event {E[X | F] is a boundary point of I}; hence here the claim is trivial. Hence #E r = Pp [y \in T] \geq Pp [Fx 1, x 2, x 3] \cdot p \land (1 - p) L > 0. 15.6 Multidimensional Central Limit Theorem .. Letting A = B \in I, we obtain the 0-1 law for invariant events: P[A] \in {0, 1}. For n \in N, define a probability Proof Let X measure $P\{-n,-n+1,...\} \in M1 \in \{-n,-n+1,...\} \in M1 \in \{-n,-n+1,...\}$ by)* $-n \in A-n$, $X - n+1 \in A-n+1$, . The random variable X1 is a proposal for the value of Y. Denote by wn = 1 + nk=1 rk the total number of balls of a given color after n balls of that color have been drawn already ($n \in N0$). = P|B|IfX|andY|are independent and N0,1 -distributed, then $\sqrt{D\sqrt{1-t}}$ = Bt, Bt X, 1 - tY. Rather, we consider empirical distributions of independent random variables with values in a finite set Σ , which often is called an alphabet. To this end, fill in the details in the following sketch. The derivative is f (x) = -xe-x, whence lim $\varepsilon - 1$ P[N $\varepsilon \ge 2$] = $-\alpha f(0) = 0$. (20.8) Remark 20.25 Sometimes the mixing property of (20.8) is called strongly mixing, in contrast with a weakly mixing system (Ω , A, P, τ), for which we require only (1 - P[A] = 0 n $\rightarrow \infty$ n n-1 lim for all A, $B \in A$. At this point, we could, for example, assume that Ye ~ U[0,1] is uniformly distributed on [0, 1]. In order for the notion of σ -additivity to make sense, the underlying class of sets must be closed under countable set operations; that is, it must be a σ -algebra. Theorem 21.18 (Strong Markov property) Brownian motion B with distributions (Px)x ∈ R has the strong Markov property. ◆ Example 17.27 (Yule process) We consider an example that resembles the preceding one at first glance. , UN } ⊂ 1 UD1 of B1 and define C1 := N i=1 U i ∩ Kn . As a first step, we define conditional independence formally (see [25, Chapter 7.3]). i=0 Then) * Px Xt0 ∈ A0 , . 98 4 The Integral Since fn ↑ f ≥ g, we have Bnc ↑ Ω for any c > 0. , Xn defines a backwards martingale. We now want to interpret X as the market price of a stock and VT as the payment of a financial derivative on X, a socalled contingent claim or, briefly, claim. Hence we have $N(x, y) + N(y, z) := m + n : m \in N(x, y)$, $n \in N(y, z) \subset N(x, z)$. We call $\psi(p) := P[$ there exists an infinite open cluster] ($\{ \#C \ p \ (x) = \infty \} = P \ x \in \mathbb{Z} d$ the probability of percolation. For $n \in N$, define $kn = \alpha n!$. If (ii) holds, then the distributions of the limiting random variables $X\lambda$ are uniquely determined and by what we have shown already, $X\lambda = \lambda$, X^* is one D possible choice. 3 Proof Take $P\mu = \mu(dx) \kappa(x, \cdot)$. Then XY also takes only finitely many values and thus XY \in L1 (P). Show that the probability measure μ has a continuous distribution function and that μ is singular to the Lebesgue measure λ . (0,1] Exercise 7.4.2 Let $n \in N$ and p, qe [0, 1]., Xn] = PXn [X ∈ A] = Pπ [X ∈ A]. Proof This is left as an exercise. We refer to analysis books like [37] where Vitali sets are used in order to show that the Lebesgue measure cannot be defined on 2R. ♦ 17.2 Discrete Markov Chains: Examples 403 Examples allows overlapping generations. Then $\lim \phi(t) = \infty = t \rightarrow \infty \lim \phi(t)$. Hence, by the superposition principle, $f(x) = \Gamma(r)$. (1.2) With this notation, $1A* = \lim \inf 1An n
\rightarrow \infty$ and $1A* = \lim \sup q(y) = 0$. $\gamma(0, \infty, t \ zr - 1 \ exp(-z) \ dz = \Gamma(r)$. (1.2) With this notation, $1A* = \lim \inf 1An n \rightarrow \infty$ and $1A* = \lim \sup q(y) = 0$. 1An . The set of polynomials with rational coefficients is countable and by the Weierstraß theorem, it is dense in any (C([0, n]), · ∞); hence it is dense in (Ω, d). In order to model this, we randomly destroy a certain fraction 1 - p of the tubes (with p ∈ [0, 1] a parameter) and keep the others. , X(An)) and let φl be the characteristic function of X(Al) for l = 1, Similarly, we get $J(x) \ge I(x)$. Reflection Why have we restricted ourselves to aperiodic Markov chains? Then v has a density w.r.t. $\mu \rightleftharpoons dv$ In this case, $d\mu$ is A-measurable and finite μ -a.e. derivative of v with respect to μ . Klenke, Probability Theory, Universitext, 213 214 9 Martingales Example 9.4 Let I = N0 and let (Yn, $n \in N$) be a family of i.i.d. Rad1/2 -random variables on a probability space (Ω , F, P); that is, random variables with P[Yn = 1] = P[Yn = -1] = 1. If (i) and (ii) hold, then $\phi = e\psi$ is a CFP. Hence we have to develop a calculus to determine the distributions of, for example, sums of random variables. Choose a ti with d(s, ti) < δ . i=n Then, we have tn -1 = u and tm = t. Definition Let $(Xn)n \in N$ be a sequence of real random variables in L1 (P) and 5.12 n let Sn = i=1 (Xi - E[Xi]). That Then E1 [τn] = n-1 k=1 k 2. Definition 17.37 A discrete Markov chain is called • irreducible if F (x, y) > 0 for all $x, y \in E$. dt Note that f0 (t) = 1 for all $t \ge 0$ and hence f1 (t) = 1 - f1 (t). , 2n - 2. 19.3 and 19.4 for illustrations.) That is, we replace C(x, y) by ∞ , C(x, y) = C(x, y), (B \leftrightarrow B c) = Then Reff n n c Bn with Bn), and thus 1 4(2n+1) if x, $y \in \partial Bn$ for some $n \in N$, else. Now, consider the case where $\lambda := 3d \min p(0, x) : x \in \{-1, 0, 1\}d > 0$. Then, by Chebyshev's inequality (Theorem 5.11), $\sigma_i = 0$, $\sigma_i = 0$, times τ (i). Hence the definition in (iv) makes sense and we have Cov[X, Y] = E[XY] - E[X] E[Y]. Let (Xn)n \in I be a square integrable F-martingale (that is, E[Xn2] < ∞ for all $n \in I$). Theorem 1.41 (Carathéodory) Let $A \subset 2\Omega$ be a ring and let μ be a σ -finite premeasure on A. $n \rightarrow \infty$ The cases of the limes superior and the Cesàro limits are similar. .) \in B $x,y \in E =$) * En 1AE 1{XN = x} pn-N (x, y)Py [B] . Since each ω is continuous, Yn is a countable supremum Yn = and is hence A-measurable. For simplicity, assume that for all x be independent and exponentially distributed with parameter $\theta > 0$. Proof Let f : E \rightarrow R be bounded and harmonic; hence pf = f . 3.3.5] for d ≥ 2). Since each ω is continuous, Yn is a countable supremum Yn = and is hence A-measurable. For simplicity, assume that for all l = 1, (ii) Assume that A1, A2, ..., X(1) = Yn, Yn-1 + Yn, . We have F(1) = x0\alpha F(x) > 0 and $F(x) = x - \alpha F(1)$ for all x > 0. \in A with AN $\uparrow \Omega$ and $\mu(AN) < measurable$ for any $N \in N$. A map $\kappa : \Omega 1 \times A2 \rightarrow [0, \infty]$ is called a (σ -)finite transition kernel (from $\Omega 1$ to $\Omega 2$) if: (i) $\omega 1 \rightarrow \kappa(\omega 1, A2)$ is A1 -measurable for any $A2 \in A2$. $1 - 2\alpha\gamma - \beta$ For $\omega \in Bn$, we conclude that (21.10) holds. \$ 5.4 Speed of Convergence in the Strong LLN In the weak law of large numbers, we had a statement on the speed of convergence (Theorem 5.14). Theorem 24.2 Let τv be the vague topology on M(E). Example 23.20 We consider the Weiss ferromagnet. m=1 (In the third step, we could change the order of summation since all summands are nonnegative.) Letting $\delta \downarrow 0$, we infer by the Borel-Cantelli lemma) * Tkn – E Tkn = 0 lim n \rightarrow \infty kn almost surely. (ii) (Closedness under complements) By definition, $A \in M(\mu *) \iff Ac \in M(\mu *)$. with probability $1 - \varepsilon$ makes a jump according to p. If $W \subset V$, then the orthogonal complement of W is the following linear subspace of $V: W \perp := v \in V:$)v, $w^* = 0$ for all $w \in W$., ωn] as the event where the outcome of the first experiment is $\omega 1$, the outcome of the first experiment is $\omega 1$, the outcome of the second experiment is $\omega 1$, the outcome of the first experiment is $\omega 1$, the outcome of the first experiment is $\omega 1$, the outcome of the first experiment is $\omega 1$, the outcome of the first experiment is $\omega 1$. C(x1) Reff $(x1 \leftrightarrow A0)$ (19.10) Definition 19.23 We denote the escape probability of x1 by) * pF $(x1) = Px1 \tau x1 = \infty = 1 - F(x1, x1)$. Then (Berp) $\otimes N \perp$ (Berq) $\otimes N$ (ii) (iii) ∞ n=1 ∞ E[Yn] converges. (7.12) First consider the case q = 1 and f \in L1 (μ). Show that the topology of weak convergence is not metrizable in general. (v) For t \leq T, let Yt := E[XT Ft]. In particular, consider a graph (E, K) and a subgraph (E almost everywhere convergence and convergence in measure do not coincide. ; ; $n \rightarrow \infty$ (iv) For every $\mu \in M1$ (E), we have ; $\mu pn - \pi$; T V $\rightarrow 0$. Then #A1 = #A2 = #A3 = 36; hence P[A1] = P[A2] = P[A3] = 16., $\omega i - 1$) and is given by a stochastic kernel ki from $\Omega 0 \times \cdots \times \Omega i - 1$ to Ωi . \blacklozenge Combining the last example with Theorem 1.53, we have shown the following theorem. Letting $x \to \infty$, we obtain $\alpha m \ge \alpha n$. It is intuitively clear that τK should be a stopping time since we can determine by observation up to time t whether { $\tau K \le t$ } occurs. Further, let $\lambda \in M1(\Sigma)$ be a distribution that is understood as the a priori distribution of this particle if the influence of energy could be neglected., xn] = P Xj $({x_j})_j = 1 = n n$ $* P X_j - 1$ $({x_j}) = P[X_j = x_j], j = 1 j = 1 and P[X_j = x_j] = px_j$. be uncorrelated random variables in L2 (P) with V := supn N Var[Xn] < ∞ . 241 243 254 12 Backwards Martingales and Exchangeability ... Without loss of generality, assume T = 1. 21.1 Continuous Versions 517 Proof (i) This is obvious since $|t - s|\gamma \leq |t - s|\gamma$ for all $s, t \in I$ with $|t - s| \le 1$. Delete the loop at the right-hand side (left in Fig. If μ is finite, then we also have (iv) \Rightarrow (iii). The left-hand side in (7.2) does not decrease if we replace f and g by |f| and |g|. In the first eight chapters, we lay the foundations that will be needed in all the subsequent chapters. , AN \in A. By Theorem 15.15 and Lemma 15.12(ii), we infer that ψ is the characteristic function of the measure ν with $\nu(A) = 12 \delta 0$ (A) + 12 $\mu(A/2) 15.2$ Characteristic Functions: Examples 343 for $A \subset R$. By Jensen's inequality, for every $\nu \in M1(\Sigma)$, $\Lambda(t) = \log e \nu(dy) \nu(\{y\}) = \log e \nu(dy) \nu(dy) \nu(\{y\}) = \log e \nu(dy) \nu($ $(\{y\}) = \mu(\{y\})e)t, y^* - \Lambda(t)$. Hence $(Xn2 \ n \in N0$ is a submartingale., Xn be integrable real random variables with $P[(X1, ..., Compute the invariant distribution and the exponential rate of convergence. (i) The map <math>(\omega, t) \rightarrow Xt (\omega)$ is measurable with respect to $F \otimes B([0, \infty)) - B(E)$. 17.6 Invariant Distributions Case 1: x = z. Then we deduce basic statements such as Fatou's lemma. The detailed version of this concise statement is: Let X1, Let B = (Bt, t ≥ 0) be a Brownian motion., d - 1, then A0, . Here we need that E is Polish since clearly every singleton is weakly compact but is tight only under additional assumptions; for example, if E is Polish (see Lemma 13.5). 2 l=1 Proof For every x \in R, we have $|eit x - 1 - itx| \le t 2x 2 2$. Theorem 6.27 (Continuity lemma) Let (E, d) be a metric space, $x0 \in E$ and let $f: \Omega \times E \to R$ be a map with the following properties. Define An = Bi for all $n \in N$. have $\mu(\Omega) < \infty$, by Theorem 6.17, there exists a sequence an $\uparrow \infty$ with sup $f \in F(|f| - an) + d\mu < 2-n$. Since μ is monotone and σ -additive, we infer $\mu(A) = \infty$ n=1 $\mu(A \cap Bn) \le \infty$ n=1 $\mu(A \cap Bn) = 0$ (n=1) (can be shown that the random variable Z := X/Y is infinitely divisible (see [65] or [131]). Let A = Zn+1 > Zn for all $n \in N0$ denote the event where Z flees directly to ∞ and let $\tau z = \inf\{n \in N0 : Zn \ge z\}$. In this case, the value of the limit does not depend on the choice of t, and the Riemann integral of f 108 4 The Integral is defined as (see, e.g., [149]) $b f(x) dx := \lim Ltn(f) = \lim Unt(f)$. (1.16) 1.4 Measurable Maps 39 The maps $R \rightarrow Z, x \rightarrow x!$ and $x \rightarrow "x\#$ are B(R) - 2Z -measurable since for all $k \in Z$ the preimages $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\} = [k, k+1)$ and $\{x \in R : x! = k\}$ and $x \to x!$ and x! and $x \to x!$ and $x \to x!$ and $x \to x$ version can be obtained as for Brownian motion by employing the fourth moments, which for normal random variables can be computed from the variances (compare
Theorem 21.9). (iii) There is a map $h \in L1$ (μ), $h \ge 0$, such that $|f(\cdot, x)| \le h \mu$ -a.e. for all $x \in E$. Let d[~] be a metric that induces convergence in measure (see Theorem 6.7). Furthermore, Z is irreducible. Further, consider sequences $t = (t n)n \in N$ of partitions t n = (tin)i=0,...,n of I (i.e., a = t0n < t1n < .3b (vi) ϕ is almost everywhere differentiable and $\phi(b) - \phi(a) = a D + \phi(x) dx$ for $a, b \in I \circ .$ (viii) Let Ω be an arbitrary nonempty set. 48 1 Basic Measure Theory is A – B(R)-measurable. Hence, it is enough to show that lim infn $\rightarrow \infty$ Sn /n > 0 almost surely. Successively, we get the nth derivative F (n) (λ) = E[(-X)n $e-\lambda X$]. Reflection Find an example of two stochastic processes that are modifications of each other but that are not indistinguishable. Use Exercise 15.4.5 to infer the statement of the central limit theorem (compare Theorem 15.38) n $\rightarrow \infty$ PSn* $\rightarrow N0,1$ weakly. $\delta 2 (\log 2)1 + 2\epsilon n = 1$) * The Borel-Cantelli lemma then gives P lim supn $\rightarrow \infty A\delta n = 0$ and hence (5.14). $\delta n = 0$ $\lim \text{supn} \rightarrow \infty \text{ Sn} = \infty \text{ almost surely. Let } \text{Ft} := \sigma (Y1, . \text{ Define }) \text{ it } X^* \text{ it } x \ 0 \ \psi 0 \ (t) = \log \text{E} \ e = e - 1 \ \nu(\text{dx}). \in M(\text{E}). (iii) \text{ Let } \text{A be an algebra. } t \rightarrow \infty \text{ Define the map } h: (0, \infty) \rightarrow (0, 1) \text{ by } h(x) = 1 - 1 - e - x \ . \in \text{A with } \text{An } \downarrow \text{ A and } \mu(\text{A1}) < \infty. (i) \text{ If } \text{I is countable, then } N := N^- \text{ is measurable and } P[N] \leq t \in \text{I } P[Nt] = 0. \text{ We conclude that } Poi \\ \lambda + Poi \\ \mu = Poi \\ \lambda + \mu \ . \end{pmatrix}$ Yn) for $n \in N$. (17.7) This follows just as in Corollary 17.10. By making ϵx smaller (if necessary), one can assume that the closure of this ball is contained in U . , d - 1, and each $A \in I$ is a union of certain Ai 's. 21. Note that P[pN] = p-s and that (pN, $p \in P$) is independent. Takeaways Properly rescaled sums of i.i.d. centred random variables with second moments converge to a normally distributed random variable. Furthermore, pt (x, y) = $e - \lambda t e^{y}$ (x, y) = e + q(x, y). Show that Cov[X1, X2] ≥ -1 Var[X1]. \blacklozenge Example 2.18 Let E be a finite set and let p = (pe) $e \in E$ be a probability vector. Let P be the Lebesgue measure on Ω . Any Kn possesses a finite covering with sets from U; hence $Kn \in C$. Hint: First show that for any $\varepsilon > 0$ and $\delta > 0$ the set $Uf\delta_{\varepsilon} := x \in \Omega 1$: there are $y, z \in B\varepsilon(x)$ with d2 (f (y), f (z)) > \delta is open (where $B\varepsilon(x) = \{y \in \Omega 1 : d1(x, y) < \varepsilon\}$). By Theorem 8.14(ii) and (viii), we get $E[|X| \land N F] \uparrow E[|X| F]$ for $N \to \infty$. (v) The class of finite unions of arbitrary (also unbounded) intervals is an algebra on $\Omega = R$ (but is not a σ -algebra). This implies that τw is the trace of the weak* -topology on Mf (E). 23.4 Varadhan's Lemma and Free Energy Assume that $(\mu \epsilon) \epsilon > 0$ is a family of probability measures that satisfies an LDP with rate function I . 16.2 Stable Distributions 387 Definition 16.26 (Domain of attraction) Let $\mu \epsilon M1$ (R) be nontrivial. Proof "(i) \Rightarrow (ii)" This implies that τw is the trace of the weak* -topology on Mf (E). 23.4 Varadhan's Lemma and Free Energy Assume that $(\mu \epsilon) \epsilon > 0$ is a family of probability measures that satisfies an LDP with rate function I . 16.2 Stable Distributions 387 Definition 16.26 (Domain of attraction) Let $\mu \epsilon M1$ (R) be nontrivial. is evident. Therefore, dx (m + n + kdy) for every $k \ge ny$; hence dx dy. Let Ef := g : g is a simple function with $\mu(g = 0) < \infty$ and let $E + f := g \in Ef : g \ge 0$. Also check that the families $\{[-\infty, a], a \in Q\}$, $\{[b, \infty], b \in Q\}$ are generators of B(R). Obviously, $A \in \sigma(\infty m = n + 1 Am)$; hence A is independent of F. Klenke, Probability Theory, Universitext, 85 86 3 Generating Functions Theorem 3.2 (i) ψX is continuously differentiable on (n) (0, 1). Then, for t \in [0, T], 2 *) 2 arc sin t/T. If E is locally compact, then (M(E), τv) is a Hausdorff space. In other words, the net flow is I (x0) + I (x1) = 0. (Dominated convergence) Assume $Y \in L1$ (P), $Y \ge 0$ and (Xn) $n \in N$ is a sequence of random variables with $|Xn| \le Y$ for $n \in N$ and such that $n \to \infty Xn \to Xa$. Then lim E[Xn | F] = E[X | F] a.s. and in L1 (P). Hence A2L \subset A2L, 0. Show that $X\sigma \ge E[X\tau | F\sigma]$. 8.1 Elementary Conditional Probabilities Example 8.1 We throw a die and consider the events $A := \{the face shows an odd number\}$ $B := \{the face shows three or smaller\}. \cap stable \lambda - system 8 \ 1 \ Basic Measure Theory \ Reflection Where does the proof of Theorem 1.19 \ fail if E is not \cap - stable? We compute the covariance function \ \Gamma \ of X, \ \Gamma \ (s, t) = Cov[Ss - sB1 , Bt - tB1] = Cov[Bs - sB1 , Bt - tB1] = Cov[Bs , Bt] - t \ Cov[B1 , B1] + st \ Cov[B1 , B1] = min(s, t) - st - st + st = min(s, t) - st + st = min(s, t) + st \ Cov[B1 , B1] = min(s, t) - st + st = min(s, t) + st \ Cov[B1 , B1] = min(s, t) - st + st = min(s, t) + st \ Cov[B1 , B1] = min(s, t) + st \ Cov[B$ min(s, t) - st. Then PX =: N μ , σ 2 is called the Gaussian normal distribution with parameters μ and σ 2. Clearly, Q± 0 P; hence the Radon-Nikodym theorem (Corollary 7.34) yields the existence of F -measurable densities Y ± such that Y ± dP = E[Y ± 1A]. Then show that μ *N0, ϵ is absolutely continuous with density f ϵ , which converges pointwise to f (as $\varepsilon \rightarrow 0$). Then A := {A $\subseteq \Omega$: A or Ac is finite} is an algebra. Since $p \leq 1.2$ (in the profitable casinos). Example 5.15 (Weierstraß's approximation theorem) Let $f:[0, 1] \rightarrow R$ be a continuous map. On the other hand, we have $(Xn+1, Yn+1) = R^n ((Xn, Yn))$. For a formal description of this model, let (In)n \in N and (Nn)n \in N an -1 19.5 Network Reduction 485 Using (19.15) we can use the values to compute u(x): P = u(x) = 29 24 + 17 5 24 - 6 2 \cdot 29 24 = 13. set For uncountable; hence neither is the composition X τ always measurable. (18.6) This decomposition is unique up to cyclic permutations. y, we also get F By the strong Markov property (Theorem 17.14), we have $\lceil \tau x 1 - 1 \text{ Ey} [] | 1{Xn = y}] = 1 + \text{Ey} [1{Xn = y}] = 1
+ \text{Ey} [1{Xn = y}] = 1 + \text{Ey}$ following, let Kn = n1/4. fdd The converse statement in the preceding theorem does not hold. Example 2.6 (Euler's prime number formula) defined by the Dirichlet series $\zeta(s) := \infty$ n-s The Riemann zeta function is for $s \in (1, \infty)$. This, however, is true if and only if ∞ n=0 (1 - pn) = ∞ . However, by a clever choice of the ONB (bn) n \in N , we can construct X directly as a continuous process. + Xn,n. This implies X ~ PPP μ . Clearly, Hy \subset A. Thus, if the number of individuals of type A in the current generation is $k \in \{0, . n=1 \text{ Then } h > 0 \text{ almost everywhere and } 3 \propto h d\mu = n=1 \mu(An) 2-n 1+\mu(A \leq 1. By assumption, q = p-1; hence x0 = y q and thus 1 + f(x0) = y q - y 1/(p-1)y = 0.$ N and Hence $11 + \lim \sup Rn =$. The preimage X-1 (A) := {X-1 (A) := {X-1} (A) := {X-1 (A) := {X-1 (A) := {X-1} coupling) Let μ , $\mu 1$, $\mu 2$, . Then Z is adapted to F. The main point of this proof consists in finding a candidate for a weak limit point for the family F. Since f and hz are continuous, for any $z \in E$, there exists an open neighborhood Uz z with hz (y) $\leq f(y) + \varepsilon$ for all $y \in Uz$. Let $\infty \propto \infty r = r s s r T \propto n=1$ Tn and $T \propto = n=1$ Tn. As (H $\cdot X$) is a martingale, the representation problem for martingales is thus reduced to the problem of representing an integrable v := YT as v0 + (H · X)T, where v0 = E[YT]. Thus (X, \tilde{Y}) is positive recurrent (hence, then the) invariant distribution of (X, in particular, recurrent) by Theorem 17.52. By Lemma 7.15, we have fg1 = |f| · |g| dµ ≤ = 1 p |f| p dµ + 1 q |g|q dµ 1 1 + = 1 = f p · gq. In fact, we can even take a different set of summands for every n. For the other inclusion, consider the class of sets A0 := A ∈ $\sigma(E)$: X-1 (A) ∈ $\sigma(X-1(E))$. Proof Let ν be a (possibly different) σ -finite measure on (Ω , A) such that $\mu(E) = \nu(E)$ for every E ∈ E. See [93]. For practical purposes, however, this is often the most interesting question. In the first section, we start with products of measurable spaces, (iii) Let (Xn)n \in Z be real-valued and stationary and let $k \in N$ and c0... To this end, we need two lemmas that ensure that the distance function associated with two measurable maps is again measurable. The process X is called a random walk on Rd with initial value x. Theorem 5.27 (Source coding theorem) Let $p = (pe)e \in E$ be a probability distribution on the finite alphabet E. Then the following statements are equivalent: Lp (i) There is an $f \in Lp$ (μ) with fn $\rightarrow f$. (iii) (Linearity) If α , $\beta \in R$, then $\alpha f + \beta g \in L1$ (μ) and ($\alpha f + \beta g d\mu = \alpha f d\mu + \beta g d\mu$. (13.6) Indeed, if f is continuous at $x \in E$, then for any $\delta > 0$, there is an $\varepsilon(\delta) > 0$ with f (B $\varepsilon(\delta)$ (x)) \subset B δ (f (x)). As we will see, we can define probabilities on σ -algebras in a consistent way. Indeed, $\mu\lambda$ is infinitely divisible with $\mu\lambda = \mu * n \lambda/n$. be independent, square integrable, centered random variables. Then Xn := Y1 + . 2.2 Independent Random Variables 61 (iv) This is trivial, as (2.7) has to be checked only for $J \subset I$ with $\#(J \cap Ik) \leq 1$ for any $k \in K$. Theorem 13.16 (Portemanteau) Let E be a metric space and let μ , $\mu 1$, $\mu 2$, L1 (P), then we define Theorem 8.14 (Properties of the conditional expectation) Let (Ω, A, P) and let X be as above. By the maximal-ergodic lemma (applied to X ϵ), we have E X0 ϵ 1 for any $k \in K$. Theorem 13.16 (Portemanteau) Let E be a metric space and let μ , $\mu 1$, $\mu 2$, L1 (P), then we define Theorem 8.14 (Properties of the conditional expectation) Let (Ω, A, P) and let X be as above. By the maximal-ergodic lemma (applied to X ϵ), we have E X0 ϵ 1 for any $k \in K$. Theorem 13.16 (Portemanteau) Let E be a metric space and let μ , $\mu 1$, $\mu 2$, L1 (P), then we define Theorem 8.14 (Properties of the conditional expectation) Let (Ω, A, P) and let X be as above. By the maximal-ergodic lemma (applied to X ϵ), we have E X0 ϵ 1 for any $k \in K$. $E[E[X0|I] 1F] - \epsilon P[F] = -\epsilon P[F]$. In Rockafellar's book [146], continuity follows from Theorem 10.1, and the statements of Corollary 7.8 follow from Theorem 12.1 and Theorem this is a discrete version of Gauß's integral theorem for (wI). Klenke, Probability Theory, Universitext, 191 192 8 Conditional Expectations By assigning the points in $\Omega \setminus B$ probability zero (since they are impossible if B has occurred), we can extend PB to a measure on Ω : P[C |B] := PB [C $\cap B$] = In this way, we get P[A|B] = #(C $\cap B$) #B for $C \subset \Omega$. For $n \ge (2 \# \Sigma)/\epsilon$, $n \to \infty$ we have $En \cap B\epsilon$ (ν) = \emptyset and hence there exists a sequence $\nu n \to \nu$ with $\nu n \in 23.3$ Sanov's Theorem 601 $En \cap A$ for large $n \in N$. Define U = u + 1 bi (t - u) = 0 and hence there exists a sequence $\nu n \to \nu$ with $\nu n \in 23.3$ Sanov's Theorem 11.19 was simple due to the assumption of finite variance of the offspring distribution. Define U = u + 1 bi (t - u) = 0 and hence there exists a sequence $\nu n \to \nu$ with $\nu n \in 23.3$ Sanov's Theorem 11.19 was simple due to the assumption of finite variance of the offspring distribution. $d\mu < \infty$, then also f $d\mu < \infty$ and f + $d\mu < \infty$. A discrete Markov chain X with D transition matrix p is a random walk on Z. (17.3) Ex f ((Xt + s)t \in A] = Px [Xt + s \in A]Fs] = EXs [f (X)] := EI 17.1 Definitions and Construction 395 Proof " \leftarrow " The time-homogeneous Markov property follows by (17.3) with the function f (y) = 1A (y(t)) since PXs [Xt \in A] = Px [Xt + s \in A]Fs] = κt (Xs, A). If A is only a semiring, then there exists an $n \in N$ and mutually disjoint sets C1, . ' ($\infty \infty$ Xn = E[Xn]. Hence, E[Xt Fs] = Xs and X is a martingale. Apart from the fact that we still have to show © The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A., xd) if, for any i = 1, . , yd)* $\in E: x_1 = 0, y_1 \ge 0$ be the set of all edges in Zd that either connect two points from $\{0\} \times Zd-1$ or one point of $\{1\} \times Zd-1$. \bullet Theorem 17.61 Let n1, n2 $\in N$ and p1, p2 $\in (0, 1)$. Let $E:=M1(\Sigma)$. Then, for every $n \in N0$ and a > 0, r + P sup $Xm \ge a \le 2P[Xn \ge a] - P[Xn = a]$. $tn \in I$, $(Xt_1, Then there exists a B \in En$ such that A = X-1 (B). • 0 Exercise 21.2.2 Let B be a Brownian motion. Proof Apply the triangle inequality in order to check (i) and (ii) in the preceding theorem. Although it is rather a matter of taste, I hope that this setup helps to motivate the reader throughout the measure-theoretical chapters. Choose (kl) \in N such that ∞ |=1 P[A Akl] < ∞ , hence $l \rightarrow \infty$ 1Akl $\rightarrow 1$ A almost surely. Compute E[X1 \land X2 |X1]. (ii) Let R := X2 + Y2 and Θ = arctan(Y/X). Lemma 14.13 Let A \in A1 \otimes A2 measurable map. For $\epsilon > 0$, choose $g \in L1$ (μ) such that sup f $\in F$ ($|f| - g \epsilon$) + $d\mu \leq \epsilon$. Definition 2.41 We say that percolation occurs if there exists an infinitely large open cluster. Consider the stochastic kernel kt (x, dy) := CPoi(x/t) exp1/t (dy). Compute the distribution of the almost sure limit limn $\rightarrow \infty$ Xn., tn) for all x \in RJ (where J = {j1, . Then also μ := ∞ n=1 α n μ n is a measure (premeasure, content). Therefore, $\lambda(A \setminus C) < \varepsilon$. Consider the event A := {UNa, b < ∞ } \cap sup{|Xt| : t \in Q+ \cap [0, N]} < ∞ ., kn, n \in N) be an independent null array of real random variables. The main goal of this chapter is to show that an arbitrary σ -finite measure ν on a measurable space (Ω , A) can be decomposed into a part that is singular to the σ -finite measure ν on a measurable space (Ω , A) can be decomposed into a part that is singular to the σ -finite measure μ and a part that has a density with respect to μ (Lebesque's decomposition theorem, Theorem 7.33). 72 2 Independence Corollary 2.39 Let (Xn)n \in N be an independent family of R-valued random variables. Fix differences T⁻tK + s - Ts n \in N. The statement for D - ϕ follows similarly. (21.23) m=1 holds and for f, g \in H)f, g* = ∞ m=1 Now consider an i.i.d. sequence (ξ n)n \in N of N0,1 -random variables on some probability space (Ω , A, P). (4.5) & Exercise 4.1.3 Let 1 \leq p $and let <math>\mu$ be σ -finite but not finite. Thus we can compute α as $\alpha = \lim u(t)/t$. (iii) By assumption, $n-1 = |h|n = (h|n E[|X|n], n \to \infty \to 0$. Then the identity map $X : \Omega
\to [0, 1]$ is a random variable on $(\Omega 1, A 1, \mu)$ that is uniformly distributed on [0, 1]. Hence we have $\mu(\{d(fn, f) > \epsilon\})$ $- \rightarrow 0$; that is, fn $- \rightarrow f$. This candidate will be constructed first as a content on a countable class of sets. 19.1 Harmonic Functions. Then we get (compare Exercise 13.1.8) M(E) = $Q \mu \in M(E) : \mu(U) < \infty$., Ifn)-1 (A) : $n \in N$; f1, . are i.i.d. with E[Y1] = 0, could be considered a fair game consisting of consecutive rounds. Example 2.24 Let X1, . Clearly, A3L \uparrow {N \geq 3} for L $\rightarrow \infty$. p q 7.2 Inequalities and the Fischer-Riesz Theorem 7.17 (Minkowski's inequality) For p \in [1, ∞] and f, g \in Lp (μ), f + gp \leq f p + gp. Remark 12.12 A backwards martingale is always uniformly integrable. Then there exists a regular conditional distribution κ Y, F of Y given F. Corollary 6.21 Let $\mu(\Omega) < \infty$ and p > 1. $= \lim_{n \to \infty} 2 k = 1$ pn, k = 0 for k > n. Hence θ (p) = P[Ac] = 0. + X(n-1) = Yn . By Dirichlet's pigeonhole principle (recall that E is finite), we can choose $\omega 1 \in E$ such that $[\omega 1] \cap Bn = \emptyset$ for infinitely many $n \in N$. The inequality holds for all y > x if and only if $t \le D + \phi(x)$. Further, let $E[|X|] < \infty$ and let $F \subset A$ be a σ -algebra. be independent N0 -valued random variables. , n - 1, we get $u(k) = \text{Reff}(0 \leftrightarrow k)$. Let $u := \max(Dn \cap [0, s])$. 21.3 Strong Markov Property 529 Conclude that $P[\tau < \infty] = 1 \land e - 2ba$. Then $\phi 1 + \phi 2 T$ $V = \phi 1 (\Omega +) - \phi 1 (\Omega -) + \phi 2 (\Omega -) = \phi 1 T V + \phi 2 T V.$ For $g \in C$, we have $\psi(1) = 1$; hence $1 \in F \cdot 2$ The partial order defined by the comparison of the distribution $V = \phi 1 (\Omega -) + \phi 2 (\Omega -) = \phi 1 T V + \phi 2 T V.$ For $g \in C$, we have g(x) = g(x + 2Kn) for all $x \in Rd$ and $n \in Zd$. functions is called (lower) orthant order. hence, in particular, $P[T \propto \infty 410 \ 17 \ \text{Markov Chains r} and T \ \text{s}$ are independent and have densities (since T r and T s have Note that $T \propto \infty 11 \ \text{r} = T \ \text{s}$] = 0. If in (5.10) the left-hand side is finite, then we can subtract the 5.3 Strong Law of Large Numbers 133 right-hand side from the left-hand side and obtain H $(p) + pe \log(qe) = e \in E pe \log(qe/pe) = e \in E pe \log(qe/pe) = e : pe > 0 \le qe - pe e \log 1 + pe pe e : pe > 0 \le qe - pe qe - pe \le 0$. Hence fn - f $\infty \rightarrow 0$. Then H n is continuous, adapted and bounded by H ∞ . A Example 1.43 Let $\Omega = Z$ and $E = En : n \in Z$ where $En = (-\infty, n] \cap Z$. More precisely, let T $\sim Poi\lambda$ and let X1, X2, Let $(Xn, Yn)n \in N0$ be a successful coupling. Here mass can emigrate but not immigrate. If p is reversible (Equation (18.8)), then $f \rightarrow pf$ defines a symmetric linear operator on L2 (E, π) (exercise!)., $\omega n \in E$, $n \in N$ } by $\mu([\omega 1, .19.6 \text{ Random Walk in a Random Walk in a$ Together with (15.9), we conclude (ii). \blacklozenge For infinitely divisible distributions on R, we would like to obtain a description similar to that in the preceding theorem. Part (i) applied to the bounded stopping times $\tau \land m \ge \sigma \land n$ yields $X \sigma \land n \ge E[X \tau \land m \ge \sigma \land n$ yields $X \sigma \land n \ge E[X \tau \land m \ge \sigma \land n]$. Comparing this to Theorem 13.16(vi), we see that the functional analysis notion of weak convergence is stronger than ours in Definition 13.12. 2π a 21.3 Strong Markov Property 531 As an application of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the reflection principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits zero. 0 Then μF is the extension of the principle we derive Paul Lévy's arcsine law [107, page 216] for the last time a Brownian motion visits and hz (z) = f (z) for all $z \in E$. We assume $Bn = \emptyset$ for all $n \in N$ in order to get a contradiction. n2 n=1 3∞) *) * $\sqrt{Proof By Theorem 4.26}$, E Yn2 = 0 P Yn2 > t dt. Fix $\varepsilon > 0$. For $n \in N$, let F-n = En and 1 Xi. Show that X has the Markov property if and only if, for every $t \in I$, the σ -algebras $F \le t$ and $F \ge t$ are independent given σ (Xt) (compare Definition 12.20). Let s, t \in D with $|s - t| \leq 2 - n0$. For any $B \in \delta(E)$ define $DB := \{A \in \delta(E) : A \cap B \in \delta(E)\}$. Clearly, we have $X1 \leq X2$ almost surely; that is, $P[(X1, X2) \in L] = 1$. Then there exists a probability space (Ω, A, P) and an independent family of random variables (Xi) i=1,...,n on (Ω, A, P) with $PXi = \mu i$ for each i = 1, ..., As F If F is continuous at x, then for every $\varepsilon > 0$, there exist numbers $q - q + \epsilon (q -) \ge F(x) - \varepsilon$ and $F(q +) \le F(x) + \varepsilon$. If E is countable, then X is called a discrete Markov process. However, N is not measurable with respect to the tail σ -algebra. Hence A \ B \in DE. Definition 9.9 (Filtration) Let $F = (Ft, t \in I)$ be a family of σ -algebras with $Ft \subset F$ for all $t \in I$. We conclude $\{\tau + \sigma \leq t\} \in Ft$. Then for any $\varepsilon > 0$ with $f(\varepsilon) > 0$, the Markov inequality holds, * E[f(|X|)]., $An \in Bb(E)$. We use this to conclude a central limit theorem for multi-dimensional independent and identically distributed random variables...] = 1{X \in A}. This is a stopping time since X^* is predictable. Therefore, we almost surely have $B \in Ft$. Hy. P[A|F] dP If F is generated by pairwise disjoint sets B1, B2, For example, is $sup{Xt, t \in [0, 1]}$ measurable? = Poiu(A). Then (E, E) is a Borel space. 4 300 13 Convergence of Measures Exercise 13.3.2 Let $L \subset R \times (0, \infty)$ and let $F = {N\mu, \sigma 2 : (\mu, \sigma 2) \in L}$ be a family of normal distributions with parameters in L. Hence F and A are independent and thus F is independent of any sub- σ -algebra of A. In this case, we write PPP $\mu := PX \in M1$ (M(E)) and say that X is a PPP μ . If r is irrational, then τr is ergodic (Example 20.9). 17.5 Application: Recurrence and Transience of Random Walks . If κ is a stochastic3 kernel from E to E and if f is measurable and bounded, then we define $\kappa f(x) = \kappa(x, dy) f$ (y). Theorem 6.12 (Fast convergence) Let (E, d) be a separable metric space. Let f, f1, f2, $\cdot \in E$ such that $\infty n=1$ $\Omega n = \Omega$ and $\mu(\Omega n) < \infty$ for all $n \in N$. Now $A \in I \subset \sigma$ (X1, X2, $\cdot Proof$ By Lemma 1.50, $M(\mu *)$ is an algebra and hence a π -system. Now we want to show that there is exactly one. We now come to the concept of stochastic order. Definition 1.35 (Continuity of contents) Let μ be a content on the ring $A \in A \infty$ with $A \subset i=1$ Ai and $\lambda n \infty$ Ai $A < \epsilon/2$. For all $\epsilon > 0$, by assumption, $\nu(A) < \epsilon$; hence we have * M0) P M $\infty = 0 = 1 - d$. By Bienaymé's formula (Theorem 5.7), we obtain + , n 1 V Var Sn = n-2 Var [Xi] < . We thus deal with balls of k different colors and with Ni balls of the ith color. • 14.2 Finite Products and Transition Kernels 309 Remark 14.18 Assume that f is almost everywhere defined and measurable (with null set N) and takes values in R. In the strong law of large numbers, however, we did not. Proof First note that the statement holds for indicator functions. In particular, μ* is a measure on M(μ*). 19.11). Hence the total number of infinite open clusters decreases by at least one. We change the Markov chain by adjoining a cemetery state Δ. Now every Markov process with countable time set (here all positive rational linear combinations of 1, t1, . , λN (listed according to the corresponding multiplicity) are real and have modulus at most 1 since p is stochastic. In particular, E[Zn] = mn for all $n \in N$. 3 3 $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn
$\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \in N$. 3 S $n \rightarrow \infty$ Then Sn $- \rightarrow S$ a.s. but limn $\rightarrow \infty$ Sn dP < S dP = 1 since S = 1 a.s. By Fatou's lemma, this is possible only if there is no integrable minorant for the sequence (Sn $n \rightarrow \infty$ and problem if E is a very large set and if sums of the type $x \in E$ f (x) $\pi(x)$ have to be approximated numerically by the estimator n-1 ni=1 f (Yi) (see Example 5.21). 635 635 644 648 657 659 26 Stochastic Differential Equations . Then under (Berr) \otimes N the family (Xn) $n \in N$ is independent and Bernoulli-distributed with parameter r (see Example 2.18). 15.2 Characteristic Functions: Examples 341 Example 15.16 Let $n \in N$, and assume that the points 0 = a0 < a1 < . Reff $(1 \leftrightarrow 0) + Reff (1 \leftrightarrow 2)$ The total current flow is I $(\{2\}) = u(1) C(0, 1) = 1 = R(0, 1) + R(1, 2) 11 C(0, 1) = 1 = R(0, 1) + R(1, 2) + R(1$ Re D 1 – $\lambda \phi(t) [-\pi,\pi)$ dt. In particular (n = 1), exchangeable random variables are identically distributed. Clearly, $\mu \in F$ is uniformly equicontinuous. Let Cb (E; C) denote the Banach space of continuous, bounded, complex-valued functions on E equipped with the supremum norm $f \propto = \sup\{|f(x)| : x \in E\}$., in $\in I$ are pairwise distinct and j1, . More generally, for a to [0, 1] is a probability measure on (0, 1], B(R) [0,1] measurable $A \in B(R)$, we call the restriction λ the Lebesgue measure on (0, 1], B(R) [0,1] measurable $A \in B(R)$, we call the restriction λ the Lebesgue measure on A. disjoint sets A1, A2, While solving the linear equations is a simple job for a computer, network reduction can give insights into the structure of the problem and can lead to general formulas also for similar networks. Proof We only have to show uniqueness of the decomposition. Then N Xn has the characteristic function $\phi Y(t) = fN(\phi X(t))$. $\in A$ of A such that $\mu(A) \ge \infty$ $\mu(Bi) - \varepsilon/2$. Before we formulate it, we state one more lemma. k=1 (iii) For any $A \in M(\mu *)$, there are sets $A - A + \varepsilon \sigma$ (A) with $A - \varepsilon A \subset A + \varepsilon A + \varepsilon \sigma$ (A) with $A - \varepsilon A \subset A + \varepsilon A +$

show linearity, it is enough to check the following three properties. The class $A := \{A \cap B : B \in A\} \subset 2A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \subset 2A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \subset 2A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \subset 2A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \subset 2A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \subset 2A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \land (1.6)$ is called the trace of A on A or the restriction of A to A.) * In particular, this holds if $E \in A \land (1.6)$ is called the trace of A on A or the restriction of A to A. theoretical education can skip in particular the first and fourth chapters and might wish only to look up this or that. Proof Without loss of generality, we may assume $\mu(\Omega) < \infty$. Since $g\epsilon/3 \ge \alpha h \downarrow \emptyset$ for $\alpha \to \infty$, for sufficiently large $\alpha = \alpha(\epsilon)$, we have $\{g\epsilon/3 \ge \alpha h \downarrow \emptyset$ for $\alpha \to \infty$, for sufficiently large $\alpha = \alpha(\epsilon)$, we have $\{g\epsilon/3 \ge \alpha h \downarrow \emptyset$ for $\alpha \to \infty$, for sufficiently large $\alpha = \alpha(\epsilon)$, we have $\{g\epsilon/3 \ge \alpha h \downarrow \emptyset$ for $\alpha \to \infty$, for sufficiently large $\alpha = \alpha(\epsilon)$, we have $\{g\epsilon/3 \ge \alpha h \downarrow \emptyset \}$ $\delta(\varepsilon)$ and any $f \in F$, $g\varepsilon/3 \ d\mu \{|f| > g\varepsilon/3 \ d\mu \{|f|$ stays in 0. Hence a representation of disjoint union of sets in A is ∞ n=1 ∞ (A1 \ An) \in A. In particular, there are more exercises and a lot more illustrations. Kolmogorov's inequality yields $\infty \infty$) * P A $\delta n \leq \delta - 2$ (l(kn)) $- 2 V kn = n = 1 n = 1 \infty V n - 1 - 2\epsilon < \infty$. This, however, is exactly (P5). , Xn . \in A such that $E \subset \infty$ En and $n = 1 \infty \mu(En) \leq \mu * (E) + \epsilon$. (ii) By (ii) and Lemma 14.11, Z E, R is a π -system that generates A. As d is complete, there is an $x \in Q$ with $\{x\} = \infty \infty i=1$ B ϵ i (ri). Thus A := $\infty n=1$ An \in Ai for every i \in I. You can supplement this book with 'Measure, Integral and Martingales' by Rene. If (fn) $n\in N$ is uniformly continuous and (fn) $n\in N$ converges to f uniformly on compact sets; that $n \rightarrow \infty$ is, for every compact set $K \subset E$, we have sups $\in K$ [fn (s) -f(s)] $\rightarrow 0$. The chapters on measure theory do not come as a block at the beginning (although they are written such that this would be possible; that is, independent of the probabilistic chapters) but are rather interlaced with probabilistic chapters that are designed to display the power of the abstract concepts in the more intuitive world of probability theory. $\epsilon \rightarrow 0$ Example 23.10 Let X1, X2, . H can be chosen to be monotone increasing and convex. Finally, let Xn := n $\xi k Bk$; k=0 that is, Xn (t) = $\xi 0 t + n$ Ak sin(k π t). \bigstar 544 21 Brownian Motion 21.6 The Space C([0, ∞)) Are functionals that depend on the whole path of a Brownian motion measurable? If (Ei)i \in I is independent, then E i \in Ik i k \in K is also independent., n = P [x1, . k $\rightarrow \infty$ [f - fnk | - g + dµ + lim sup gk dµ $\leq \varepsilon$, k $\rightarrow \infty$ Corollary 6.26 (Lebesgue's convergence theorem, dominated convergence) Let n $\rightarrow \infty$ f be measurable and let (fn)n \in N be a sequence in L1 (µ) with fn \rightarrow f in measure. 20.4 Application: Recurrence of Random Walks 505 n -> ∞ (i) Sn' -> ∞ almost surely. Other classes of functions that are often considered are convex functions or indicator functions or indicator functions that are often considered are convex functions of Random Variables 259 Clearly, gk (0) = N - M - l, where l = #{i $\leq k : xi = 0$ }. If A \in A and if there exists a null set N such that E(ω) holds for every $\omega \in A \setminus N$, then we say that E holds almost everywhere on A. Let V = F : R \rightarrow R is right continuous, monotone increasing and bounded be the set of distribution functions of finite measures on R. k Therefore, lim n $\rightarrow\infty$ Now Rn \leq $1 + 1 \max Sk n k = 1, \dots, n = \lim n \to \infty$ - Theorem 20.19, this implies P[A] = 0, BtN) (21.17) * $\rightarrow EB\tau f$ (Bt1, . Then f is measurable (and everywhere defined). A further tool will be the convergence theorem for backwards martingales that will be formulated in Sect. Let P be an arbitrary finite measurable partition of Ω . Exercise 2.2.1 Let X and Y be independent random variables with X ~ exp θ and Y ~ exp θ and Y ~ exp θ and Y ~ exp θ and I + X be uniformly distributed on A (see Example 1.75). 106 4 The Integral The probability of no win until the nth game is (1 - p)n; hence P[Sn = 1 -2n] = (1 - p)n and P[Sn = 1] = 1 - (1 - p)n. Since F (y, x) > 0, there exists an l \in N with pl (y, x) > 0. We come to an extension theorem 1.41). MORE FROM QUESTIONSANSWERED.NET Showing 1-38 Start your review of Probability Theory: A Comprehensive Course It is a tough book so if you are studying probability theory (and measure theory) for the first time don't read this book alone. for all $z \in [0, 1]$. t = 0 for every t $\in [0, 1]$. Then PX =: N μ , Σ is the d-dimensional normal distribution with parameters μ and Σ . Hence E[X | F] dP = P[Bi] E[X | Bi] = E[1Bi X] = X dP. , An $\subset E$, we have $P[Xi \in Ai | A] = \Xi \infty$ (Ai) for all i = 1, .92 3 Generating Functions Let T, X1, X2, .7.3 investigate the case p = 2 in more detail. Without loss of generality, we assume that if X is irreducible and recurrent, then an invariant measure of X is unique up to a multiplicative factor. Takeaways For vector-valued random variables to converge, it is enough that the projections to one-dimensional subspaces converge, it is enough that the projections to one-dimensional subspaces converge (Cramér-Wold). n=1 $\propto \mu(A)$. Theorem 21.9 There exists a probability space (Ω , A, P) and a Brownian motion B on (Ω , A, P). k=1 Let $c \geq 0.2$ (iii) Let X be log normally distributed (see Example 15.5). $\leq \epsilon t + 2 \epsilon - 2 E Xn$, Hence, for all $\epsilon > 0$, lim sup mn $\leq \epsilon$, $n \rightarrow \infty$ and the Central Limit Theorem and thus lim mn = 0. ϵ Recall the definition of a distribution function function function Electrical Networks We consider symmetric simple random walk on Z2. We have used this inequality in order to give an (almost sharp) upper bound on the speed of convergence in the strong law of large numbers (Theorem 5.29). Then E is a π -system and σ (E) = 2 Ω . For each i \in I, let (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , A) \rightarrow (Ω i , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi :
(Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable space and let Xi : (Ω , Ai) be a measurable sp a random variable with generated σ algebra σ (Xi) = Xi-1 (Ai). Furthermore, Xe \leq Xe for any $e \in E$. + xN = M, we have) * 1 P X1 = x1 , . Thus formally we can build the factor space. 19.17, determine the probability Pa [$\tau z < \tau a$]. De Finetti's structural theorem says that an infinite family of E-valued exchangeable random variables can be described by a two-stage experiment. In order also to obtain these results for submartingales, in the first section, we start with a decomposition theorem for adapted processes. + Xn. Furthermore, for simplicity, the individuals are assumed to be haploid; that is, cells bear only one copy of each chromosome (like certain protozoans do) and not two copies (as in mammals). + km, then *) P Mn,1 = k1, The eigenvalues are $\lambda \epsilon_k = (1 - \epsilon)\lambda k + \epsilon$, k = 0, Manifestly, no state x with $x \equiv 0$ and $x \equiv 1$ is stable. \blacklozenge © The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. Remark 14.10 Every ZJ is a σ -algebra, and Z and Z*R are algebras. In contrast to Chap. 1 - P[Zi = 0] = 22i2 For i, $n \in N$, define Xi := Yi + Zi and Sn = X1 + . Furthermore, the statement analogous to (21.10) holds not only for s, $t \in [0, 1]$. + Tns , $n \in N$ } be the jump times of (Rt) and (St). (21.20) k=-\infty Use (21.19) and (21.20) (compare also (21.38)) to conclude that $Px[\tau > T] = 4 \pi \infty k = 0.1 2k + 1$ $2\pi 2 T (2k+1)\pi x \exp - (2k+1)\pi x$ binary prefix code. (ii) Give an example of two distribution functions F and G on R2 such that $(x, y) \rightarrow F(x) \land G(y)$ is not a distribution function on R4. 21.3 for a computer simulation of Xn, n = 0, 1, 4, 64, 8192. Define B1 = A1 n-1 and Bn = An \ i=1 Ai \in A for n = 2, 3, . We construct a finite covering Vx1, . However, if we change the time set, then the assumptions have to be strengthened: If (Xt)t \in Rd is a process with values in E, and if, for certain α , $\beta > 0$, all T > 0 and some C < ∞ , we have d+ β E[(Xt , Xs) α] \leq C t - s2 for all s, t \in [-T , T]d , (21.13) then for every $\gamma \in (0, \beta/\alpha)$, there is a locally Hölder- γ -continuous version of X. Here the best prediction for X is its mean; hence E[X] = E[X |F] as shown in (vi). Hence f can be expanded in a Fourier series $f(x) = \infty$ cn e2 π in x for P-a.a. x., n}: |Sk| \geq t. k=l+1 Hence $\phi\mu = \phi$., else. A similar statement holds for the case where r = r1 = r2 = . To do this, open your device and run a search for either a specific online directory, such as Telkom or WhitePages. and that are uniformly distributed on [0, 1]. All of these properties are direct consequences of the corresponding properties of the integral., $\omega(tk)$ is continuous. Hint: Show (iv) first for step functions (see Exercise 4.2.6). 50 1 Basic Measure Theory (iii) Let a > 0 and let Caua be the distribution on R with density $x \rightarrow 1.1$. and thus $P[N] \leq P[R \ c] + P[N]$ We come to the main theorem of this section. Then there exists a successful coupling (X, Y). N i=1 (17.12) i=0 Takeaways A Markov process indexed by the natural numbers is called a Markov chain. Definition 1.21 (Borel σ -algebra) Let (Ω , τ) be a topological space. Consider the random variables Yn := X(An) and Y = X(A). (8.15) Therefore, by (8.14), F[~] (\cdot , ω) is a distribution function for any $\omega \in \Omega \setminus \mathbb{N}$. 1.2 Set Functions .. A stochastic process $X = (Xt) t \in I$ is called stationary if $L[(Xt + s) t \in I] = L[(Xt) t \in I] = L[(Xt) t \in I]$ for all $s \in I$. Let Bn (ε) = $\{d(f, fn) > \varepsilon\}$ and $B(\varepsilon) = \{d(f, fn) > \varepsilon\}$ and $B(\varepsilon$ $(\{k\}) - \mu(\{k\}) < \epsilon 4 k = 0$ for all $n \ge n0$., An $\in B(E)$ be pairwise disjoint. P[Ns = k, Nt - Ns = 1] = e k! !! (5.21) This implies that Ns and (Nt - Ns) are independent. (i) E[|X|] $\le \lim \inf n \to \infty E[|Xn|]$. Manifestly, $\beta(A) = \mu * (A)$ for any open A. 17.1 (page 412). In fact, in order that μ be a measure (not only a signed measure), we still have to show that all of the masses $\mu(\{x\})$ are nonnegative. At the stock exchanges, not only are stocks traded but also derivatives on stocks. 2 2 0=x \rightarrow 0 x 2 n \rightarrow ∞ Lemma 15.49 If (i) of Theorem 15.44 holds, then $\nu n \rightarrow \delta 0$ weakly. (iii) If F is uniformly integrable and if, for any g \in G, there exists an f \in F with $|g| \leq |f|$, then G is also uniformly integrable. (iii) If $\Omega 0 = R$ and I = N, then RN is the space of sequences ($\omega(n)$, $n \in N$) in R. Now define $f := u/u^2$. By the principle of conservation of energy, the last term equals 2 u(x) - u(y) D(x, y) = 4D(A1)(u1 - u0) = 0. However, since the chain is irreducible, for every $y \in E$, there exist numbers i(y). The Kolmogorov-Sinai theorem shows that the entropy that was introduced in Definition 20.30 for simple shifts coincides with the entropy of Definitions and Construction In the following, E is always a Polish space with Borel σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra on Ω = E N0 . 17.1 Definitions and Construction In the following, E is always a Polish space with Borel σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by take P = {e} × E N , e \in E} which generates the product σ -algebra by t an arbitrary index set and let (Ai)i \in I be a family of events. Theorem 15.9 (Characteristic function) A finite measure $\mu \in Mf(Rd)$ is characteristic function. The events Fx 1 ,x 2 ,x 3 and Gy,x 1 ,x 2 ,x 3 are independent, and if both of them occur, then y is a trifurcation point. Show that lim $r \downarrow 0 \mu(x + rC) = f(x) r d \lambda d(C)$ for $\lambda d - 1$ almost all $x \in Rd$. -r - kr (vii) The negative binomial distribution br, $p(\{k\}) = k(-1)p(1-p)k$, $k \in N0$, - = with parameters r > 0 and $p \in (0, 1)$, is infinitely divisible with br, p - *n. n=1 Proof (i) Note that $A \cup B = A(B \setminus A)$. Then we have $L[(Xi1, ... \cup C . + Xm + E[Xm+1] + ... Since F(y, x) > 0$, there exists a $k \in N$ with pk(y, x) > 0. l=1 Theorem 3.7 (Poisson approximation) Under the above assumptions, the distributions (PS n)n \in N converge weakly to the Poisson distribution Poi λ . 1.1 Classes of Sets . i=1 Hence μ is σ -additive and therefore a premeasure. That is, it is measurable with respect to σ (D1 , . Let Sn = ni=1 Hi Di = (H · X)n be the gain after n rounds. i=0"(see Exercise 20.5.1). Example 19.31 Symmetric simple random walk on Z3 is transient. There an example of [164] (see also [165, 166]) for orthogonal series is developed further. Let σ and τ be bounded stopping times with $\sigma \leq \tau$. an If μ is stable (in the broader sense) with index $\alpha \in (0, 2]$, then PX is said to be in the domain of normal attraction if we can choose an = $n1/\alpha$. be independent ZD -valued random variables with P[Yi = x] = p2 (0, x). Let $n \in N$ and A1, Then there is a measurable space (Ω, A) and a Markov process $(Xt) t \in I$, $(Px)x \in E$, $A \in B(E)$, $t \in I$. $(Xt - x, t \in [0, \infty)$ is a Brownian motion (with X Takeaways A continuous stochastic process can be considered as a random variable with values in the space C([0, ∞)) of continuous functions. (12.2) 12.1 Exchangeable Families of Random Variables 261 In particular, 1 $\phi(X)$. By the continuous functions. (12.2) 12.1 Exchangeable Families of Random Variables 261 In particular, 1 $\phi(X)$. $-1 \rightarrow P \circ \phi -1$
; hence $Pn \rightarrow P$. In fact, the entropy of a delta distribution is zero and for a distribution on n points, the maximal entropy is achieved by the uniform distribution and equals log(n) (see Exercise 5.3.3). Use the reflection principle to show that, for every $x \in (0, a)$, ∞ $Px[\tau > T] =)*(-1)n Px BT \in [na, (n + 1)a]$. t Henceforth we assume that the limit q(x, y) exists for all y = x and that $q(x, y) < \infty$ for all $x \in E$. If A2L,0 occurs and if we open all edges in BL, then at least two of the infinite open clusters get connected by edges in BL. The fifth section, we prove a multidimensional version of the CLT. l=1 By Theorem 5.35, the random variables Mn,T, 1, . Furthermore, $\phi(k)$ exists in ($-\varepsilon, \varepsilon$) and is continuous on ($-\varepsilon, \varepsilon$) for 15.4 Characteristic Functions and Moments 353 any k = 0, . Lemma 3.10 ψ n = ψ Zn for all n \in N. Reflection In Etemadi's theorem, we assumed that the random variables X1, X2, This beautiful theorem could be shown using the tools that we developed in the previous sections for other purposes. We abbreviate LipK (E) := Lip(E; R). For the time being, however, this theorem gives us enough examples of interesting families of independent random variables. Clearly, it is the smallest σ -algebra that contains E. Remark 12.7 If $A \in \sigma$ (Xn, $n \in N$) is an event, then there is a measurable $B \subset E N$ with $A = \{X \in B\}$. $r \ge t$, $r \in I$ Hence $N^- \subset R \subset \cup N \neq I$ and $Y_- = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = \Omega \times \Omega^- \subset R \subset \cup N \neq I$ and $Y_- = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = \Omega \times \Omega^- \subset R \subset \cup N \neq I$ and $Y_- = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If we let $\Omega = 0.316$ 14 Probability Measures on Product Spaces If $\Omega = 0.316$ 14 Probability Measures on Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Probability Measures O Product Spaces If $\Omega = 0.316$ 14 Pro Let X be a Markov chain with transition matrix p such that there ; ; $n \rightarrow \infty$ exists a successful coupling. By the upcrossing inequality (Lemma 11.3), for every N > 0 and every finite set I \subset [0, N], we have E[UIa,b] \leq (E[XN |] + |a|)/(b - a). In the following, let (E, τ) be a topological space with Borel σ -algebra E = B(E) := σ (τ) and with complete metric d. There exists a finite subcovering $U = \{Ut1, ..., Xn \text{ for some n. Define } \sigma_n := \inf\{m \in N : Sm+n = Sn\}$, $Bn := \{\sigma n < \infty\}$ for $n \in N0$ and ∞ B := Bn. Takeaways In this section, we have encountered the most prominent integral inequality. (For the measurability for convex functions, Hölder's inequality and Minkowski's inequality.) of the integral see Exercise 21.1.2.) (ii) Show that almost surely λ {t : Bt = 0} = 0. As for sums of independent random variables, we first show convergence for a fixed time point to the distribution of a certain limiting process. Theorem 17.40 (Pólya [134]) Symmetric simple random walk on ZD is recurrent if and only if D ≤ 2. On the other hand, for $|x| \ge 1$, we have $|ft(x)| \le |eit x| + 1 + |t/x| \le 2 + |t|$. Takeaways We have got acquainted to the notions stochastic process, filtration, adapted, stopping time, and σ -algebra of τ -past. (4.2) n \in N α 1 i=1 i Ai for some α 1, Exercise 19.1.1 Let p be the substochastic E \times E matrix that is given by p(x, y) = p(x, y), $x, y \in E$ (with p^{-1} as in (19.2)). Therefore, E[|Xi| Fi-1] = |Xi-1|, if |Xi-1| = 0, Assume that A is chosen so that $Px [\tau A < \infty] = 1$ for every $x \in E$. 3.2 Poisson Approximation 89 For $k \in N0$, the kth derivative is $f(k)(0) = \alpha(\alpha - 1) \cdots (\alpha - k + 1)$. Define the map $h: R \to [0, \infty)$ by $h(x) = 1 - \sin(x)/x$ for x = 0and h(0) = 0. Fix such a J and let $j \in I \setminus J$. < i=1 th and $\alpha 1$, . Then $\infty \infty \mu(\{f \ge n\}) \le f d\mu \le \mu(\{f \ge n\}) \le h = h$ denoted by $M\sigma(\Omega, A)$. \bullet Exercise 21.4.2 (Martingale convergence theorems) Let X be a stochastic process with RCLL paths. 3 f du = γ). be independent, square integrable random vari Let 1 ables with ∞ Var[X n] < ∞ . Assume that we have a sequence of events A1, A2, . That is, we have to show that $x \to \kappa(x, A)$ is measurable with respect to B(E) $B(E)\otimes I$. Theorem 11.19 Let $Var[X1,1] \in (0, \infty)$. Theorem 24.7 The distribution PX of a random measure X is characterized by its char matrix p. Theorem 1.4 Assume that A is (0, 1). $= \infty n=1 xn 2$ the dyadic expansion (with lim supn $\rightarrow \infty xn = 1$ for definiteness). (16.4) n ϕ_n , (t) exists and that Assume that A is (0, 1). $= \infty n=1 xn 2$ the dyadic expansion (with lim supn $\rightarrow \infty xn = 1$ for definiteness). (16.4) n ϕ_n , (t) exists and that Assume that A is (0, 1). $= \infty n=1 xn 2$ the dyadic expansion (with lim supn $\rightarrow \infty xn = 1$ for definiteness). (16.4) n ϕ_n , (t) exists and that Assume that A is (0, 1). (Lemma 15.12). Let $Bn = \{-n, . As the new and the old types of the replaced individual are independent, as a model for the gene frequencies, we obtain a Markov chain X on E = \{0, N1, . n \rightarrow \infty If E is locally compact and Polish, then in addition each of the following is equivalent to the previous statements. If E is locally compact and Polish, then in addition each of the following is equivalent to the previous statements.$ integrable real random variable whose distribution PX has a density f (with respect to the Lebesgue measure λ). We say that the sequence (fn) $n \in N$ converges in mean to f, symbolically L1 fn $\rightarrow f$, $n \rightarrow \infty$ if fn $-f 1 \rightarrow 0$. Recall that de Finetti's theorem states that there exists a random probability measure Ξ on E such that, given Ξ , the random variables X1, X2, . • Remark 13.14 (i) In functional analysis the notion of weak convergence is somewhat different. However, this is not the case! If we thicken the great circle slightly such that its longitudes range from Θ to $\Theta + \varepsilon$ (for a small ε), on the equator it is thicken (measured in meters) than at the poles., 6} × B. 19.6 Random Walk in a Random Environment 491 We are now in the position to prove a theorem of Solomon [158]. \bullet Takeaways The measure extension theorem shows how to extend contents from semirings to σ -algebras but usually does not give a concrete construction. Here α , $\beta < -1$ and $\in [0, 1]$. Then there is a sequence E1, E2, . i=1 Then ST \in L2 (P) and Var[ST] = E[X1] 2 Var[T] + E[T] Var[X1]. Hence there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all x, $y \in [0, 1]$ with $|x - y| < \delta$. E is called a generator of $\sigma(E)$. Let $d(x) := y \in E$ (y - x) p(x, y) for $x \in E$. In the following, let (E, τ) be a topological space with the Borel σ -algebra E = B(E) (compare Definitions 1.20 and 1.21). $y \in E y \in E$) * Thus (since $Px \tau x 1 = 0$ $= 0 \ \mu x \ p(\{x\}) = \infty$ ' (Px tx1 = n + 1 = 1 = $\mu x \ (\{x\})$. This example was a bit extreme. Let $A = \{Y > Y\} \in F$. Now, is the code constructed above optimal, or are there codes with smaller mean length? + kD = n., $\alpha n \in R$, we obtain 1 f (s) dWs = 0 n αi Wti – Wti-1., fk \in Cb (E) and F (x1, . (iii) Compute the expectation and variance of 1 0 2 1 Bt + 1 = 1 = $\mu x \ (\{x\})$. Bs ds dt. Let $\phi 1$, $\phi 2 \in M \pm .$ Since A is a semiring, for every $n \in N$ there is an m n k mn $\in N$ and sets Cn1, . I0 For $k \in N$, let *) $\psi k(t) = \log E$ eit (Xk $-\alpha k$) = it x $e - 1 - itx \nu(dx)$. The following theorem shows that properties (P1)-(P5) characterize the random variables (NI, I $\in I$) uniquely and that they form a Poisson process. Then (fn) $n \in N$ converges in measure. However, by the definition of the trifuccation point x, this is
impossible. 22.1 Iterated Logarithm for the Brownian Motion ... This game can also be the price of a stock that is traded at discrete times on a stock exchange. Since $\sigma \leq \tau$, we thus get $A \cap \{\tau \leq t\} = A \cap \{\sigma \leq t\} \cap \{\tau \leq t\} \in Ft$. It is easy to see that the set $D = x \in X$. $E = R and then come to a couple of applications. Now write f(\omega 1, \omega 2) = f(\omega 1, \omega 2) \cdot h1(\omega 1)h2(\omega 2). If A is an algebra, then obviously <math>\emptyset = \Omega \setminus \Omega$ is in A. Reff $0 \leftrightarrow B(n + 1)c = B(k) \leftrightarrow B(k)c = k = 0$ ($0 \leftrightarrow \infty$) = Therefore, $Reff 1 3 \infty k = 0$ k $2 3 = 1 < \infty$. Hence, for t > s, t = E[Yr Fs] = Xs. i = 1 i = 1 Therefore, X is i.i.d. given A. Thus R is really an algebra. extension of the real line, and the inclusion $R \rightarrow R$ is measurable. Since E is Polish, A is compact. The idea is that Ft reflects the knowledge of an observer at time t. (Compare Corollary 9.34.) & Exercise 9.2.4 (Azuma's inequality) Show the following. Therefore, P + N - 1, $A0n = P[\tau < N] \ge n = 0 1$. This implies $\mu(A \cap E) = \nu(A \cap E)$ for any $A \in A$ and E \in E with $\mu(E) < \infty$. Now let (Yn,i, n \in N0, i \in N0, i N0, i \in N0, i N0, i N0, i N0, i total current is Ri n 1 I = i=1 Ri and thus we have Ceff $(0 \leftrightarrow 1) = n$ i=1 Ci and n 1 Reff $(0 \leftrightarrow 1) = Ri - 1$. n $\rightarrow \infty$ This shows (2.5). For the geometric offspring distribution, ψ (n) can be computed explicitly. As an example, we analyze the Weiss ferromagnet. This μ is called the uniform distribution on Ω . 4 1 Basic Measure Theory (ii) Let $\Omega = R$. At this point, we use the topology only to make M1 (E) a measurable space, namely with the Borel σ -algebra B(M1(E)). Define TnK := n YiK UnK := and i=1 n ZiK for n \in N. Exercise 19.6.1 Consider the situation of Theorem 19.35 but with the random walk restricted to N0. As above, we get that the probability of drawing out of n balls exactly bi balls of color i for each i = 1, . Summation over x and y yields the general case. ., Yn \in L2 (P) and α 1, . (ii) (Monotone 3 3 convergence) If fn \uparrow f, then the integrals also converge: fn dµ \uparrow f dµ. Show that E[Xn] = n-1 k=0 r + k r + s+k for any n \in N. The corresponding statement holds for T s. In metrizable spaces, the notions compact and sequentially compact coincide. The second equivalence in (ii) follows by (3.2). $\delta \downarrow 0 \ \omega \in A$ Theorem 21.40 A family (Pi, i $\in I$) of probability measures on C([0, ∞)) is weakly relatively compact if and only if the following two conditions hold. and let E be the exchangeable σ -algebra. Since $q-1 \ q = p$, we have $\kappa(f) \ p \cdot gp \geq fg1 = |f|q||1 = fq = fq \cdot fq q q-1 = fq \cdot gp$ 21.7 Convergence of Probability Measures on $C([0, \infty))$. Let $n \in N$, and assume that ϕ is 2n-times differentiable at 0 with derivative $\phi(2n)(0)$. We may assume that the sequences $l \rightarrow \infty$ (kln) $\in N$. Now consider the following rescaling: Fix $x \ge 0$, start with Z0 = Z individuals and consider Z^{\sim} tn := ntn! for $t \ge 0$. Formally, P[B] = 0 where B is the event where there is one color of which only finitely many balls are drawn. , BN $\in \beta N$. In Hilbert spaces, continuous linear maps can be represented as a scalar product with some fixed vector (Riesz-Fréchet theorem). Hence $n \rightarrow \infty$ Yn 1A $\rightarrow 1$ A P[B] almost surely. Here we employ the right continuity of the increasing function Fi that belongs to μi . Exercise 14.4.1 Assume that ($\nu t : t \ge 0$) is a continuous convolution semigroup. However, in general, Ft +s is a strict superset of Ft; hence $\tau -s$ is not a stopping time. Note that (for fixed n), (Nn,t) t =0,1,... (18.13) If (18.12) and (18.13) hold for x1, Then, for $f \in F$, $|f| d\mu \leq \{|f| > g\epsilon\} \{|f| > g$ \in {0, 1} and sk =) * P Xi = xi for any i = 1, . Thus, in fact, the network is equivalent to that in Fig. Again, the exact values can 23.4 Varadhan's Lemma [167]) Let I be a good rate function and let (με) ε>0 be a family of probability measures on E that satisfies an LDP with rate function I Hence) ' (* f1 dξn (X) ··· fk dξn (X) E f1 (X1) ··· fk (Xk) – E ' (' F dµn,k (X) = E F dνn,k (X) – E n→∞ ≤ F ∞ Rn,k – → 0. Define p(i, j) := 1 exp(-βWj) Z for all i, j = 1, . Then Hb (p) ≤ - pe logb (qe) (5.10) e ∈ E with equality if and only if Hb (p) = ∞ or q = p. are a.s. continuous, then there is a continuous martingale X with the following and $X tn n \rightarrow \infty - \rightarrow X t$ in Lp for every $t \ge 0$. Then (Xi)i \in I is independent and Xi is normally distributed with parameters (μ i , σ i 2). In particular, the unit current flow is uniquely determined. Denote by R(x, y) = 1 \in (0, ∞] C(x, y) the resistance of the connection)x, y*. 1.0 0.5 0.0 0.0 0.2 0.4 0.6 0.8 1.0 -0.5 Fig. (iii) By Theorem 1.96, any nonnegative measurable map is a monotone limit of simple functions. (iv) If $\alpha = 1$, then there exists a $b \in \mathbb{R}$ such that $\mu * \delta - b$ is stable with index α . On the other hand, for $j \in \{\tau k + 1, . (iii) \phi X(t) = \phi - X(t)$. (17.4) Proof If the conditional distributions exist, then, by Theorem 17.9, the equation (17.4) is equivalent to X being a Markov chain. Then (12.4) formally becomes k *) $\Xi \infty$ (Al)ml. Then ϕ Pnk (t) = ϕ n (t ek) is the characteristic function of Pnk. on (Ω , A, P) with distributions PX = μ and PXn = μ n, n \in N, such that $n \rightarrow \infty$ Xn $\rightarrow X$ P-a.s. Hint: Use Exercise 13.2.13. f d μ ., σ k }, we have (H ·Y)j \geq (H ·Y) τ k = (H ·Y) σ k – 1., Ck,ck \in A such that k. Hence PXn [X \in A] = P π [X \in A] almost surely and thus (with $\sigma = n$ in (20.4)) Pn [X \in A X0, . (19.11) 19.4 Recurrence and Transience 475 Theorem 19.25 We have pF (x1) = 1 Ceff (x1 $\leftrightarrow \infty$). Let k := (t + s)n!. n=0 Hence, $[\tau x1 - 1 Ey]$] 1{Xn = y}] = n=01. A Often this measure will be denoted by the same symbol λ when there is no danger of ambiguity. , Xtn) is n-dimensional normally distributed, and (iv) integrable (respectively square integrable) if X is real-valued and $E[|Xt|] < \infty$ (respectively $E[(Xt) 2] < \infty$) for all $t \in I$. To show this, let $V \subset Rn$ such that $V \in B(Rn)$. Hence also $\sigma X - 1$ (E) $C X - 1 \sigma (E) = 2$ and n4 = 4Clearly, $P[\tau - n (A \ A \varepsilon)] < \varepsilon$ and $P[\tau - n (B \ B \varepsilon)] < \varepsilon$ for 508 20 Ergodic Theory every $n \in Z$. " = " For $J \subset I$ and $j \in I \setminus J$, choose $Ej = \Omega$. If $f = m \alpha i$ 1Ai (4.1) i=1 for some $m \in N$ and for $\alpha 1$, 324 14 Probability Measures on Product Spaces This implies J - 1) (A) = $PJ \circ (XL = A \ A \ P (d(\omega 0, . For a fixed time t, on the basis of previous observations, we$ er X is already in K for the last time., n, let ki be a substochastic kernel from × Ωk, k=0 k=0 (Ωi, Ai) or from (Ωi-1, Ai-1) to (Ωi, Ai). Proof Let A ∈ A1 ⊗ A2. By assumption, $\nu(\Omega) = fZ d\mu = \nu(Ac)$; hence $\nu(A) = 0$ and thus $\nu 0 \mu$. Then the recursion $\kappa 1 \otimes \cdots \otimes \kappa i := (\kappa 1 \otimes \cdots \otimes \kappa i - 1) \otimes (\Omega 1 \otimes \cdots \otimes \kappa i)$ $Xk^2 - n(\omega) - X(k-1)^2 - n(\omega) < 2 - \gamma n$ for $k \in \{1, ..., (18.11) \ 3 \text{ Here } m(\mu) = pN(x) \ \mu(dx)$, where the probability $pN(x) = 1 - 1 - rr \mid 1 \mid x \mid 1$, N if r = 12. Replace the parallel edges with resistances R1 = 5 and R2 = 1 by one edge with R = (15 + 1) - 1 = 56 (right in Fig. (15.5) $\theta \in (0,1]$ Define a continuous function fn : $R \rightarrow [0, \infty)$ by fn (0) = 1 and fn (x) = (-1) (2n)! x n - 2n cos(x) - n - 1 k=0 x 2k (-1) (2k)! . Show that (PeX1/\epsilon) $\epsilon > 0$ satisfies an LDP with convex good rate function 2 I (x) = 1 + x arc sinh(x) - 1 + x 2 . 23.2). 4336 15 Characteristic Functions and the Central Limit Theorem Exercise 15.1.8 Let (Ω, τ) be a separable topological space that satisfies the T3 1 2 separation axiom: For any closed set $A \subset \Omega$ and any point $x \in \Omega \setminus A$, there exists a continuous time with Q-matrix $q(x, y) = p(x, y) - 1\{x=y\}$. (vi) Trivially, E[X] is measurable with respect to F. gm : Rd \rightarrow C, Let C be the algebra of finite linear combinations of the gm. Remark 14.17 Let g, h : $\Omega \rightarrow$ R be measurable finite almost everywhere. Since f \equiv 0 and since E \ A is finite, there is an x0 \in E \ A is finite, there is a zoo of strong laws of large numbers, each of which varies in the exact assumptions it makes on the underlying sequence of random variables. For $d \ge 2d-1 \le pc \le 23$ (Theorem 2.45). n $\rightarrow \infty$ Corollary 15.26 For every $\alpha \in (0, 1]$ and r > 0, $\phi \alpha, r(t) = e - |rt|$ is the characteristic function of a symmetric probability measure $\mu \alpha, r$ on R. As above, we have $A \le (0, 1]$ and r > 0, $\phi \alpha, r(t) = e - |rt|$ is the characteristic function of a symmetric probability measure $\mu \alpha, r$ on R. As above, we have $A \le (2, -1]$ and r > 0, $\phi \alpha, r(t) = e - |rt|$ is the characteristic function of a symmetric probability measure $\mu \alpha, r$ on R. As above, we have $A \le (2, -1]$ and r > 0, $\phi \alpha, r(t) = e - |rt|$ is the characteristic function of a symmetric probability measure $\mu \alpha, r = 0$. We conclude that $\mu(U) \le \mu(A) +
\infty \varepsilon 2$ n=1 Inner regularity Replacing B by B c, the outer regularity regulari n1 Sn = m(ξ n (X)). We are well aware that it is not enough to show this for the case n = 2 only. Before stating Jensen's inequality, we give a primer on the basics of convexity of sets and functions. Proof This follows from Theorem 19.25 and Rayleigh's monotonicity principle (Theorem 19.19). (23.22) Varadhan's lemma (more precisely, the tilted LDP) and Sanov's theorem are the keys to building a connection to the variational principle for the free energy. As an example where inequality holds, instead of the standard example from calculus textbooks (fn = $n \cdot 1(0, 1/n)$, f = 0), we studied a game of hazard that we will encounter in a different context later. Then P + , + , n Aj = P Bj $\in J_j \in J_j \in J_j$ $= e1 \in B^{-1} \cdots j = 1$ n $en \in B^{-n} j = 1$ pej = n j = 1 pej = n j = 1 pej $\in J$ $j \in J$ $e \in B^{-1} j$ This is true in particular for #J = 1. This behaviour has been quantified in Sect. Then there exists a modification X Proof For a, $b \in Q+$, a < b and $I \subset [0, \infty)$, let UIa, b e the number of upcrossings of (Xt) t $\in I$ over [a, b]. J = 1 pej = n j = 1. This behaviour has been quantified in Sect. Then there exists a modification X Proof For a, $b \in Q+$, a < b and $I \subset [0, \infty)$, let UIa, b e the number of upcrossings of (Xt) t $\in I$ over [a, b]. J = 1 pej = n j = 1 16.11 If ϕ is an infinitely divisible CFP, then there exists a $\gamma > 0$ with $2 \alpha |\phi(t)| \ge 12 e - \gamma t$ for all $t \in \mathbb{R}$. Let $Mn := i \in \Lambda Xn$ (i) be the total number of individuals of opinion 1 at time n., Y(d) and $\tau = max\{\tau(1), ... + X^n, we \text{ get } P[Sn \ge 0] = = \{x1 + ... + xn \ge 0\} \mu(dx1) \cdots \mu(dxn) \{x1 + ... + xn \ge 0\} - \tau x n e - \tau x1 \mu(dx^2 \mu(dx^2 1) \cdots e n) (- \pi E e - \tau E e -$ Sn 1{Sⁿ ≥0}. Note that the contour in Fig. For $q \in \{p, p\}$ and $e \in E$, we define 0 1, if $Ye \le q$, qXe := 0, else. For i = 0, 1, let $pDLi := \{Xe = i \text{ for all } e \in EL\}$. Define $\sigma n := 2-n$ "2n $\sigma #$ and $\tau n := 2-n$ "2n $\sigma #$ the time set $[0, \infty)2$ with covariance function $Cov[X(s,t), X(s,t)] = (s \land s) \cdot (t \land t)$. Furthermore, we show that the absolutely continuous part has a density. be random measures and $\lambda 1$, $\lambda 2$, . By the mean value theorem of calculus, for all $n \in N$ and for almost all $\omega \in \Omega$, there exists a yn $(\omega) \in I$ with gn $(\omega) = f(\omega, yn(\omega))$. (Ai)i \in I is an independent family of σ -algebras and if A is trivial, then (Ai)i \in I is independent given A., Xn-1. + Xn)/n of i.i.d. random variables (law of large numbers). (19.13) yields P = u(x) = Reff (0 \leftrightarrow 1) + Reff (0 \leftrightarrow x) - Reff (x \leftrightarrow 1).) of events and assume that each family is independent but they are not necessarily independent of each other., n hn, l (k) = n j = l pn (k, j) = $\left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right) \left(\begin{array}{c} 0 \end{array}\right) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \left(\begin{array}{c} 0 \end{array}\right) \left(\begin{array}{c} 0$ Theorem 7.26 (Riesz-Fréchet representation theorem) Let $(V,) \cdot , \cdot *$) be a Hilbert space and let $F: V \rightarrow R$ be a map. For $\delta > 0$, Chebyshev's inequality yields (together with (5.8)) * $\infty \infty *$) * Var Tkn -2 P Tkn -E Tkn $> \delta$ kn $\leq \delta$ kn2 n=1 n=1 = $\delta -2 \infty$ kn-2 n=1 kn Var[Ym] = \delta -2 \infty kn-2 n=1 kn Var[Ym] = $\delta -2 \infty$ kn -2∞ kn Var[Ym] * m-2 E Ym2 < ∞ by Lemma 5.20. For the second one, we must show the mutual inclusions. (i) If PX = PY, then E[X] = E[Y]. 691 Notation Index ... We want to show that ϕ is the characteristic function of a probability measure $\mu \in M1$ (R). random variables are the median, expectation and variance. $n \rightarrow \infty$ The left-hand side is $F\tau$ -measurable. 7.3). Note that the uniqueness of μn and ϕn , respectively, is by no means evident. Now assume (6.2). $k=1 \neq (2d)-1 L-d n P[Mk-1 \in \{0, Ld\}]$. \diamond The situation is note that the uniqueness of μn and ϕn , respectively, is by no means evident. Now assume (6.2). $k=1 \neq (2d)-1 L-d n P[Mk-1 \in \{0, Ld\}]$. completely satisfying as we have made the very restrictive assumption that Y is real-valued. Let (Ω, F, P) be a probability measure on (Ω, A) . that cannot all occur jointly. Define A0n := An, A1n := A \ An, and let Pn := n s(i) Ai : s \in \{0, 1\}n i=1 be the partition of A that is generated by A1, are identically distributed on Rd with characteristic function $\phi X \in E$ and numbers $\alpha 1$, $\alpha 2$, are independent and Berv(A)-distributed; hence we have $X(A) \sim Poiu(A)$ (see Theorem 15.15(iii)). \blacklozenge Example 1.44 n (Distributed; hence we have $X(A) \sim Poiu(A)$ (see Theorem 15.15(iii)). \in Rn). (ii) If X, X1 , X2 , Let (An)n \in N be a sequence in A such that An \uparrow A and let A0 = \emptyset . $\geq 1-21+2z$ It is easy to check that P0 [$\tau z < \infty$] = 1 for all $z \in$ N0. * Exercise 2.3.2 Consider two families (A1 , A3 , A5 , . The claim follows by the observation that (for $\alpha = -1/2$) we have -1/2 additional 2n - n. * Exercise 21.5.3 Define Y := (Yt)t $\in [0,1]$ by Y1 = 0 and t Yt = (1 - t)(1 - s) - 1 dWs for $t \in [0, 1)$. The situation is quite the opposite for, e.g., the Poisson distribution $\mu = Poi$ with parameter > 0 and $\nu = N0, 1$. Changing the roles of x and Then F :(y, x) > 0. The speed of convergence is thus exponential with a rate that is determined by the spectral gap $1 - |\lambda 2|$ of the second largest eigenvalue of p. By the strong law of large numbers, we thus have $1 n \rightarrow \infty$ Yi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and Laws of Large Numbers and $1 n \rightarrow \infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and 1 n $-\infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and 1 n $-\infty$ Zi (x) $- \rightarrow$ F (x) almost surely n n Fn (x) = i=1 130 5 Moments and 1 n $-\infty$ Zi (x) $- \rightarrow$ F (x) algebras of A. \blacklozenge We are particularly interested in σ -algebras that are generated by topologies. (v) If $\alpha = 1$, then dn = (c + - c -) n log(n), n \in N. If log(0) has a finite variance, this follows by the central limit theorem. As f and gx are continuous, for any $x \in E$, there exists an open neighborhood Vx x with gx (y) $\geq f(y) - \varepsilon$ for any $y \in Vx$. be i.i.d. real 2 random variables nwith $\mu := E[X1] \in R$ and $\sigma := Var[X1] \in (0, \infty)$. Then define Fn (x) = $\int \frac{1}{x \ln \beta} \left(x \ln \beta + \frac{1}{x} \ln \beta \right) \left(x \ln \beta + \frac{1}{x} \ln \beta \right) \left(x \ln \beta + \frac{1}{x} \ln \beta \right) \left(x \ln \beta + \frac{1}{x} \ln \beta \right) \left(x \ln \beta + \frac{1}{x} \ln \beta \right) \left(x \ln \beta + \frac{1}{x} \ln \beta \right) \right)$ different frequencies have different distributions., km \in N0, we have) * P Nti – Nti–1 = ki for any i = 1, . Proof (i) " \Rightarrow " Assume f = 0 almost everywhere. In fact, by the tower property of the conditional expectation (Theorem 8.14(iv)), we get *) E[Xs+2 Fs] = E E[Xs+2 Fs+1] Fs . 18.3 or 18.2, where d = 2, E0 = {1, 3, 5, 7} and E1 = {2, 0, 4, 6}, 8}. By a backward induction, this yields the claim. Proof We have) * $E[f(|X|)] \ge E f(z)$) * $\ge F(z)$ P |X| $\ge \varepsilon$. k=1 Evidently, Xn = Mn + An . Then G(0, 0) = ∞ 2n n=0 n p(1 - p) n = ∞ -1/2 n=0 n n - 4p(1 - p) . Without loss of generality, we may hence assume that E is σ -compact and thus separable. 5.1 Moments 115 Proof Assume first that X and Y take only finitely many values. 19.13 Steps 7 and 8. 2 k=1 (iv) Under the assumptions of (iii), Azuma's inequality holds: * $\lambda 2 P |Mn| \ge \lambda \le 2 exp - n 2 k=1 ck2$) for all $\lambda \ge 0$. As $x \to gx$ (y) is monotone, we get $D + \phi(x) \ge D - \phi(x) \ge D +
\phi(x)$ for x > x. For large n, the expectation describes the typical approximate value of the arithmetic mean (X1 + (3.1) n=0 \otimes The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. Hence, by symmetry, the chance of winning should be p = 18/37 < 12. Combining this with the assumption, we get $\phi(y) = \infty$ for all y > x., $\omega n \in E$ }. If she wins the second game, she leaves the casino and otherwise doubles the stake again and so on. In the sense of Definition 14.6, $X = (Xt)t \ge 0$ is thus the canonical process on (Ω, A) . If the values of f on A are prescribed, then we say that f solves a Dirichlet problem. Let $(\Omega i, Ai)$ be the measurable space of the ith experiment, i = 0, $n \rightarrow \infty$ (v) lim sup μn $(E) \leq \mu(E)$ and lim inf μ (G) $\geq \mu(G)$ for all open $G \subset E$. Theorem 1.12 (Relations between classes of sets) (i) Every σ -algebra also is a λ -system, an algebra also is a λ -system, an algebra also is a λ -system, and a σ -ring. ($\omega 1$,..., ωn) =($\omega 1$,..., ωn) =(($\omega 1$,..., ωn) =((($\omega 1$,..., ωn)) =(((\omega in measure: If f, f1, f2, . Then, for A0, A1 \in E with A0, A1 $= \emptyset$ and A0 \cap A1 $= \emptyset$, 19.3 Finite Electrical Networks 471 R1 R2 x0 = 1 x1 = 6 R3 R4 R5 u(0) = 0 u(6) = 1 R6 Fig. With (Xt)t ≥ 0 , we have almost constructed the so-called Brownian motion. Theorem 7.18 (Fischer-Riesz) (Lp (μ), \cdot p) is a Banach space for every $p \in [1, \infty]$. Let $\mu = \lambda [0,1]$ Lebesgue measure restricted to [0, 1]. Denote by A B := $(A \setminus B) \cup (B \setminus A)$ for A, B $\subset \Omega$ (1.14) the symmetric difference of the two sets A and B. (i) We say that (Xn)n $\in N$ for any $\epsilon > 0$. Proof For $x \in \sqrt{R}$, we have $x \ge 1\{|x| > \epsilon\} \le (\epsilon) - \delta |x|^2 + \delta = 0$. By the choice of k and since the increments of B are stationary, we have +,) * P B \in AN, n] $n \rightarrow \infty \le lim \sup n P[B \in AN, n, 1]$ $n \rightarrow \infty \le lim \boxtimes n \ge lim \ge$ by $(0, \infty)$, then we define the size-biased distribution P P:(A) = m-1 P x P (dx). In the case $\gamma = 1$, Hölder continuity (see Definition 13.8). By construction, the coupling (X, Y) is successful. Example 18.16 (Ising model) The Ising model (pronounced like the English word "easing") is a thermodynamical (and quantum mechanical) model for ferromagnetism in crystals. 4.1 Construction and Simple Properties 99 Theorem 4.8 Let $f: \Omega \rightarrow [0, \infty]$ be a measurable map. In the following, we always assume that (Ω, A, P, τ) is a measurable map. In the following, we always assume that (Ω, A, P, τ) is a measurable map. In the following, we always assume that (Ω, A, P, τ) is a measurable map. In the following (Nn1, T)] = ∞ [f (Nn1, T)] = ∞ [f (Nn1, T)] $P[T = t] t = 0 \le \infty$ E[f (Nn2, t)] P[T = t] = E[f (Nn2, t)]. For s, t > 0, we have $Cov[Xs, Xt] = ts \cdot Cov[B1/s, B1/t] = ts \min s - 1$, t $-1 = \min(s, t)$. Then, with the usual arguments, extend it step by step first to simple functions, then to nonnegative measurable functions, then to nonnegative measurable functions. We do not intend to go into the details and we only briefly touch upon the topic. & Exercise 15.1.4 Show that, under the assumptions of Theorem 15.1.1, Plancherel's equation holds: $\mu(\{x\})^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x \in \mathbb{Z}d$ [$-\pi,\pi$] ($\varphi \mu(t)^2 = (2\pi) - d x = d$ P Xk-1 (Ik) = Xk-1 (Ik + Nk) Fk-1 . 2 2 For these k, $|\lambda k|$ equals $\gamma := 1 - 4r(1 - r) \sin(\pi/N)$. Since $\nu \le \mu + \nu$, the linear functional $3 L2 (\Omega, A, \mu + \nu) \rightarrow R$, $h \rightarrow h d\nu$ is continuous. Let C be the algebra of finite linear combinations of elements of C. be measurable functions that converge to some f almost everywhere. Case 2: $p \in (1, \infty)$. Let Z be a connected component of HL that contains at least 2.4 Example: Percolation 83 one point $x \in TL$. However, with a little bit of abstract nonsense, one can apply a voltage of 1 and at the points to the right the voltage 0, then by symmetry no current flows through the superconductors. Then Xn := k cl Yn-l, n \in Z, l=1 defines a stationary process X that is called the moving average with weights (c1, . Hence, there is a phase transition between the high temperature phase 608 23 Large Deviations Fig., Xn)-] < ∞ and) * E ϕ (X1, . 14.3 Kolmogorov's Extension Theorem ..., Xn) > E[g(X1, By Theorem 8.29], we obtain the regular conditional distribution κY , F of the real random variable $Y = \phi \circ Y$. We will study product measures in
a systematic way in Chap., k. $n \rightarrow \infty$ (ii) μ is called upper semicontinuous if $\mu(An) - \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ in A with $\mu(An) < \infty$ (ii) μ is called upper semicontinuous if $\mu(An) - \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ in A with $\mu(An) < \infty$ (iii) μ is called upper semicontinuous if $\mu(An) - \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ (ii) μ (An) $- \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ (iii) μ (An) $- \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ (An) $- \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ (An) $- \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ (An) $- \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ (An) $- \rightarrow \mu(A)$ and An \downarrow A. 515 515 522 529 532 535 544 546 549 553 560 22 Law of the Iterated Logarithm .. 2 x,y \in Proof (Rayleigh's monotonicity principle, Theorem 19.19) Let I and I be the unit current flows from A1 to A0 with respect to C and C , respectively. * n $\rightarrow \infty$ Exercise 13.2.4 Let E = R and λ be the Lebesgue measure on R. * Theorem 19.15 An electrical potential u in (E, C) is a harmonic function on $E \setminus A$: $u(x) = y \in E 1 C(x, y) u(y) C(x)$ for all $x \in E \setminus A$. We see that the square variation process. We assume that d is a metric on Q that induces the standard topology and such that (Q, d) is complete. Proof See, e.g., [88, Theorem 3.2.18], [168, Theorem 4.17] or [156]. Fix $\varepsilon > 0$ and let $\alpha = 1 + \varepsilon$. \blacklozenge 450 18 Convergence of Markov Chains Fig. Let P = be the Bernoulli measure. 1 Hence we have Reff ($0 \leftrightarrow 2$) = -1 - 1 - 1 C(0, 1) + C(1, 2). Check that in this case A has either probability 0 or 1. - + (iii) In order to show RW = $RW = \infty$ almost surely, it is enough to show n lim supn $\rightarrow \infty$ k=0 log(k) > $-\infty$ and lim supn $\rightarrow \infty$ 1k= $-n \log(k-1) > -\infty$ almost surely if $E[\log(0)] = 0$., $zn \in \Sigma$, then the state of this ensemble can be described by x := n1 ni=1 δzi . Then (by) *) n * -Zkn)4 Ex (Z⁻ tn+s $-Z^-$ sn)4 = n-4 (tn)4 E nx! (Zk+1) * = t 4 E nx! 24Zkn + 12(Zkn)2 + 2Zkn = t 4 $26 \text{ nx!} + 24 \text{ nx!k} + \text{nx!2} \le 26x \text{ t} 3 + 24x \text{ s} \text{ t} 2 + x 2 \text{ t} 2 \le (50\text{Nx} + x 2) \text{ t} 2$. The electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the graph) minimises the electrical current between two contacts (vertices in the is given as the reciprocals of the expected return times. be i.i.d. random variables that are independent of X1, X2, Reflection Consider the random walk on the integers with transition matrix given by p(k, k + 1) = r and p(k, k - 1) = 1 - r for some $r \in [0, 1]$. Show that $\mu = -1$, x^* is characterized by its Laplace transform $L\mu(\lambda) = e\mu(dx)$, $\lambda \in [0, \infty)d$ A family C of measurable maps $E \rightarrow R$ is called a separating family for F if, for any two 278 13 Convergence of Measures $\mu, \nu \in F$, the following holds: f d $\mu = f$ d ν for all $f \in C \cap L1$ (μ) $\rightarrow L1$ (ν) $\Rightarrow \mu = \nu$. Step 4. Exercise 19.5.1 Show the validity of the star-triangle transformation., m. For $\lambda \ge 0$, show that 2 *) $E = -\lambda \tau = \exp - ba - ba 2 + 2$ 2λ ., Mn,m) are independent. Some readers might prefer to skip the somewhat technical construction of general Markov processes in Sect. 41.2 Show that in Theorem 11.14 the converse implication may fail. $n \rightarrow \infty$ C(x1) Example 19.26 Symmetric simple random walk on E = Z is recurrent. The destroyed tubes will be called "closed", the others "open". = $1 - it/\theta \ 1 + it/\theta$ of independent random variables leads to multiplication of the transforms. A process W with properties (i) and (ii) is called a Brownian sheet. We have Poia 0 Poib if and only if $\beta > 0$ or $\alpha = 0$. a 2a 2 (21.21) & 21.4 Supplement: Feller Processes In many situations, a continuous version of a process would be too much to expect, for instance, the Poisson process is generically discontinuous. By Theorem 14.31, the equations (14.19) hold. Since the support of f is contained in (-K, K)d, f^{\sim} is contained in (-K, K)d. construct Brownian motion as a continuous Gaussian process. The sets lim inf An := $n \rightarrow \infty \infty$ Am and lim sup An := $n \rightarrow \infty \infty$ Am n=1 m= $n \infty \infty \infty$ Am n=1 m= $n \infty \infty$ Am n=1 m= $n \propto \infty$ YtD). $e \in E$ e $\in E$ The length of this code for the first n symbols of our random information source is thus approximately $-nk=1 \log 2$ ($pXk(\omega)$) = $-\log 2 \pi n (\omega)$. $2n - 1 n \rightarrow \infty$ and Hn = 0 for t > T. $n \rightarrow \infty$ (iii) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iii) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iii) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iii) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iii) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iii) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge \lim_{n \to \infty} u \rightarrow \infty$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge u$) $\mu = v$ -lim μn and $\mu(E) \ge u$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge u$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge u$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge u$ (iv) $\mu = v$ -lim μn and $\mu(E) \ge u$ (iv) $\mu = v$ -lim $\mu n \rightarrow \infty$
 (iv) $\mu = v$ -lim $\mu n \rightarrow \infty$ (iv) $\mu = v$ -lim $\mu n \rightarrow \infty$ (iv) $\mu = v$ -lim $\mu n \rightarrow \infty$ (iv) there exists a pmax \in Wm that maximises the entropy; that is, that H pmax = supp \in Wm H (p). For $n \in N$, define En := σ (F : F : E N \rightarrow R is measurable and n-symmetric) and let En := X-1 (En) be the σ -algebra of events that are invariant under all permutations \in S(n). Then E[$\phi(X1 \ , . Hence the limit point Q \ is unique and thus (PXn \ n \in N \ define En \ de$ weakly to Q. \blacklozenge Remark 8.16 Let X : $\Omega \rightarrow R$ be a random variable such that $X - \in L1$ (P). Then $\phi(x) \ge t + \cdot (x - E[X]) + \phi(E[X]) = \phi(E[X]) + \phi(E[X]) = \phi(\tau, X\tau) = 2$ Thus, in the last step of (17.9), equality holds and hence also in (17.8). Convergence to Stable Distributions To complete the picture, we cite theorems from [54, Chapter XVII.5] (see also [62] and [128]) that state that only stable distributions of rescaled sums of i.i.d. random variables X1 , X2 , . For any subset A of Rn , we have B(A, d) = B(Rn , d) . Thus $f - gp \le \mu(\Omega)1/p$ f $-g\infty$ for f, $g \in L^{\infty}(\mu)$ and hence i is continuous. Hence also $A = \infty A n = 1 1/n$ is countable and thus a null set. \bullet If A and B are independent, then Ac and B also are independent since $P[A \cap B] = P[B] - P[A \cap B] = P[A \cap B$ (8.4) If P[A] > 0, then $P[\cdot |A]$ is a probability measure. $m \to \infty$ $n \ge m$. Let $n \in N$ and let A1, . The actual density could be gained using Hilbert space theory, in particular the Riesz-Fréchet representation theorem. If all k = 1 i / $\nu n = n \ pk \ \mu k$. We have thus shown that (14.6) and (14.7) hold for f = 1A for all $A \in A1 \otimes A2$. If $A \in B(Rn)$ and $f: Rn \rightarrow R$ is B * (Rn) - B(R) measurable (or $f: A \rightarrow R$ is B * (Rn) - B(R) measurable (or $f: A \rightarrow R$ is B * (Rn) - B(R) measurable), then we write $f \ d\lambda := f \ 1A \ d\lambda . < yN-1 < f \ \infty < yN$ such that $yi \in R \setminus A$ and $|yi+1 - yi| < \varepsilon$ for all i. For $L \in N$, let $BL := f \ A \ \Delta R$ is B * (Rn) - B(R) measurable (or $f: A \rightarrow R$ is B + (Rn) - B(R) measurable). $\{-L, . The printed phone books grew in popularity during the decades and centuries. Now, by Example 2.36(ii), Y = VT, then, in particular, v0 - v0 = ((H - H) · X)T = VT), then, in particular, v0 - v0 = ((H - H) · X)T = VT)$ As with continuous maps, the composition of measurable maps is again measurable. Generate Fn and let $F \leftarrow F \circ Fn$. Let un, k (x) := ξn (x) $\otimes k = n-k n$ i1 ,..., ik = 1 $\delta(xi1$,..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distribution on E k that describes k-fold independent sampling with replacement (respecting the order) from (x1 , ..., xik) be the distributio recursively by $\mu * 1 = \mu$ and $\mu * (n+1) = \mu * n * \mu$. 19.10 Steps 1 and 2. Consider a Polish space Σ that is interpreted as the space of possible states of a particle. As we required only first moments, in general, we cannot expect to get any useful statements. Now assume $d \ge 2$. l=1 $n \to \infty$ $n \to \infty$ However, for $\varepsilon > 0$, we have νn ($[-\varepsilon, \varepsilon]c$) = Ln (ε) $- \to 0$; hence $\nu n \rightarrow \delta 0$. Let S(n) be the set of permutations : {1, . Let i := - + Wi - /Wi + for i \in Z and RW and RW be defined as above. Let L2 (μ) = L2 (μ)/N be the factor space. Then T \subset E, and strict inclusion is possible. However, we will see that all martingales can be 9.4 Discrete Martingale Representation Theorem and the CRR Model 225 represented as stochastic integrals if the increments Xn+1 - Xn can take only two values (given X1, Show that $D := \{t \in R : \Lambda(t) < \infty\}$ is a nonempty interval and that Λ is infinitely often differentiable in the interior of D. Show that $D := \{t \in R : \Lambda(t) < \infty\}$ variables, then for PX almost all $x \in R P[X + Y \in \cdot |X = x] = \delta x * PY$. Define the Schauder functions by Bn,k (t) = [0,t] B C bn,k (s) $\lambda(ds) = 1[0,t]$, bn,k. Let d be a metric on Ω , and denote the open ball with radius r > 0 centered at $x \in \Omega$ by Br (x) = { $y \in \Omega : d(x, y) < r$ }. Hint: Apply Exercise 13.2.14 to the image measures $\mu \circ f - 1$. It is enough to show that there exist continuous adapted processes X n, $n \in N$, for which (25.3) holds. 0 As f is continuous and f (0) = 1, the last integral converges to 0 for $K \rightarrow \infty$. This version is called Feller's (continuous) branching diffusion. 21.3 The processes X n, n = 0, 1, 4, 64, 8192 from the Fourier Construction of Brownian motion. Hence En = ni=1 ($Ei-1 \cap I$) Ω i). Clearly, the uniform distribution on E is invariant but lim δx pn does $n \rightarrow \infty$ not exist for any $x \in E$. Definition 8.11 (Conditional expectation) A random variable Y is called a conditional expectation) A random variable Y is called a conditional expectation of X given F, symbolically E[X | F] := Y, if: (i) Y is F -measurable. Show that for t > 0 Etemadi's inequality holds: '() * P max $|Sk| \ge t \le 3 \max P |Sk| \ge 1$ t/3. In this case, $\mu \in \text{Dom}(\mu)$. (ii) This is a direct consequence of (i). Lemma 7.23 If) \cdot , * is a semi-inner product topology of the topology on V that is generated by the pseudo-metric d(x, y) =)x - y, x - y*1/2). 21.2 Construction and Path Properties 525 Clearly, X is a centered Gaussian process with continuous paths. For example, consider I = N0 and X1 = X2 = X3 = . We will see in a computer simulation that the Ising model displays this critical temperature effect. Define Xn := D1 + . The three points on the very right of the graph form a loop that can be deleted from the network without changing any of the remaining voltages. Let $P\pi = \pi({x}) Px$. 4 19.3 Finite Electrical Networks 467 Exercise 19.2.2 Let $\beta > 0$, $K \in N$ and W1, The empirical distributions of the first n random variables yield a tight family which, by Prohorov's theorem, has a limit point. 19.16, determine Ceff (a $\rightarrow z$) and Pa [$\tau z < \tau a$]. The total energy (or Hamilton function) of the system in state x is the sum of the individual energies, $H(x) = Hi(x) = i \in \Lambda$ 1{x(i)=x(j)}. Then, by Lemma 18.2, there exists an m0 \in N such that pn (x1, x2) > 0 and pn (y1, y2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 and pn (y1, y2) > 0 for all $n \ge m0$. An Determine
one such sequence (An)n \in N such that pn (x1, x2) > 0 and pn (y1, y2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. An Determine one such sequence (An)n \in N such that pn (x1, x2) > 0 for all $n \ge m0$. , A1) and let κ be a finite transition kernel from $\Omega1$ to $\Omega2$. (6.2) Theorem 6.17 The family $F \subset L1$ (μ) is uniformly integrable if and only if sup |f| d $\mu = 0$. Remark This method for generating random variables with a given distribution Q is called rejection sampling, as it can also be described as follows. We can regard the graph as an electrical network with unit resistors at each edge, voltage 0 at 0 and voltage 1 at 1. If d = gcd(n, r) and $Ai = i, \tau(i), \tau 2(i)$. Theorem 4.9 (Properties of the integral) Let $f, q \in L1$ (µ). (iii) The distribution on N with weights $c n\alpha$ if n is odd. The map $Rd \rightarrow C, x \rightarrow ei s \lambda, x^*$ is $n \rightarrow \infty$ continuous and bounded; hence we have $E[ei s)\lambda, Xn *] \rightarrow E[ei s)\lambda, Xn *]$ (In fact: Choose an arbitrary countable base U of the topology. Furthermore, by summing over $k \in N0$, this yields $Nt - Ns \sim Poi\alpha(t - s)$. Note that, in particular, this definition does not require that the individual payments be independent or identically distributed. 3 (i) We3have f = 0 almost everywhere if and only if $f d\mu = 0$. A map $f: \Omega \setminus N \to \Omega$ is called a μ -almost everywhere defined and measurable map from (Ω , A) to (Ω , A) to (Ω , A), if f -1 (A) \subset A. Proof By Lemma 14.13,3 for every $\omega 1 \in \Omega 1$, the map f $\omega 1$ is measurable with respect to A2; hence If ($\omega 2$) $\kappa(\omega 1, d\omega 2)$ is well-defined. By Helly's theorem, there is a monotone right continuous function F : R $\rightarrow [0, 1]$ k $\rightarrow \infty$ and a subsequence $(Fnk)k\in N$ of $(Fn)n\in N$ with $Fnk(x) \rightarrow F(x)$ at all points of continuity x of F. & Exercise 17.6.4 Let $r \in [0, 1]$ and let X be the Markov chain on N0 with transition matrix (see Fig. Hence it suffices to show that Nn1, T \leq st Nn2, T. (ii) If $\mu = f \lambda n$ and $\nu = g \lambda n$ are finite measures with Lebesgue densities f and g, then $\mu * \nu = (f * g)\lambda n$. Assume that (17.13) and (17.14) hold and that $\lambda := \sup |q(x, x)| < \infty$. Finally, by monotone convergence, for any $f \in F$, $H(|f(\omega)|) \mu(d\omega) = \infty$ $(|f| - an) + d\mu \le n = 1 \infty 2 - n = 1$. For $\nu \in A$, there is an $\varepsilon > 0$ with $B\varepsilon(\nu) \subset A$. 21.7 Convergence of Probability Measures on $C([0, \infty))$ 547 Next we derive a useful criterion for tightness of sets $\{Pn\} \subset M1(C([0, \infty)))$. Proof One implication has been shown already in Corollary 16.7. Hence, let ϕ be an infinitely divisible CFP. Since any Kn is bounded, we have $\lambda(Kn) < \infty$. On the other hand, every stochastic process (Xt)t \in I (on an arbitrary probability space (Ω, A, P)) with stationary independent increments defines a convolution semigroup by $\nu t = P \circ (Xt - X0) - 1$ for all t \in I. Most σ -algebras A are simply too large. Here, however, we follow a different route. Proof (i) For t \in R, h \in R \ {0} and k \in {1, . Of course, transience in the case p = 12 could also be deduced directly from the strong law of large numbers since limn $\rightarrow\infty$ n1 Xn = E0 [X1] = 2p - 1 almost surely. Hint: Use the approximation theorem for measures (Theorem 1.65) with the semiring of left open intervals to show the assertion first for measurable indicator functions. Starting from a normed vector space X of continuous linear functionals $X \rightarrow R$. Then $N := \omega \in B \setminus N$ and every $x \in V$. As $x \in [0, 1)$ increases U = 1, Define A := $\Omega + 0$ and $\varepsilon := n10$. Appealing to this intuition we define the number of clicks until time t by Nt := # {n \in N0 : Tn \leq t}, E12 from Theorem 1.23. $E \subset D$, by the π - λ theorem (Theorem 1.19), $D = \sigma$ (E) = A2 . $n = m n = m + 1 \infty$ hence $Em \downarrow \Omega + .$ Definition 4.2 Define the map I : $E + \rightarrow [0, \infty]$ by I (f) = m $\alpha i \mu(Ai)$ i=1 if f has the normal representation f = m i=1 αi 1Ai. Then we write A = B (mod μ). Then, by (2.4) (in Example 2.4), for 2.2 Independent Random Variables 63 n \in N and x \in E n , we have (x) = 1, x = 1 $\{Mn\varepsilon > 0\}$. $x \lor y = max(x, y) \land x \land y = max(x, 0) \land x = max(x, 0) \land y = max$ $: A \in Bb(E)$) the smallest σ -algebra on M(E) with respect to which all maps IA : $\mu \rightarrow \mu(A)$, $A \in Bb(E)$, are measurable. In particular, the fair price does not change if we pass to the equivalent martingale X. Now assume the claim is proved for $n \in N$. Let (U x,y,n : x, y $\in Zd$, $n \in N0$) be an independent family of (Zd) N-1 -valued x,y,n x,y,n random variables $U x_y n = (U1, . + Yn - 1, xn - 1 = xn] * x * x = PY1, 1n - 1 ({xn}) = qxn n - 1 = p(xn - 1, xn)$. almost surely., 2n - 1. (E, τ) is called metrizable if there exists a metric d on E such that τ is induced by the open balls Be (x) := { $y \in E : d(x, y) < \varepsilon$ }. We give two proofs for this statement. (ii) If X1, X2, . Lemma 5.26 (Entropy inequality) Let b and p be as above. Dominated convergence yields E X0 ϵ 1Fn \rightarrow) E X0 ϵ^* . Let X be an adapted real-valued stochastic process. Corollary 2.38 Let (An)n \in N be a sequence of independent events. By the strong law 7.4 Lebesgue's Decomposition Theorem 177 of large numbers, for any $r \in \{p, q\}$, there exists a measurable set Ar $\subset \Omega$ with (Berr) \otimes N ($\Omega \setminus Ar$) = 0 and 1 Xi (ω) = r n $\rightarrow \infty$ n n for all $\omega \in Ar$. 19.12 Steps 5 and 6. (iii) (fn)n $\in N$ is uniformly integrable and there is a measurable map f such that meas fn $\rightarrow f$., Rn that connect the same two nodes can by replaced by a single edge with resistance R = (R1-1 + . + 1 g Lp (P) 1 p 7 Lp - Spaces and the Radon-Nikodym Theorem 172 7.3 Hilbert Spaces In this section, we study the case p = 2 in more detail. Then E = N i=1 Ei and by (13.6), $\mu \partial E i \leq \mu f - 1$ ({yi-1}) + $\mu f - 1$ ({yi-1} equations with two unknowns has to be solved. Due to the lack-of-memory property of the exponential distribution, also the remaining lifetimes of the other x - 1 individuals are independent and exponentially distributed with parameter 1. If $\mu *$ is σ -subadditive, then for any $F \in U(A)$, we have $F \in F \mu(F) \ge \mu(A)$; hence $\mu(A) \ge \mu(A)$. Hence, let X = M + 1A = M + A be two such decompositions. n & Exercise 15.5.2 Let Y1, Y2, ..., 6 N, $A = (2\{1,...,6\}) \otimes N$ is the product 1 $\otimes N \sigma$ -algebra and P = is the Bernoulli measure (see Theorem 1.64). Clearly, t $\rightarrow Xt$ is continuous at every point t > 0. This theorem is a cornerstone for a functional analytic proof of the Radon-Nikodym theorem in Sect. Hence $\int \int P_{r,r} dP_{r,r} dP_{r,$ $P[Ye = q] = -p, | 1 - p, p \text{ if } q = p, if q = 1, \omega k] \cap Cn, in = \emptyset$ for all k, $n \in N$. (iv) (Monotonicity) If $X \le Y$ almost surely, then $E[X] \le E[Y]$ with equality if and only if X = Y almost surely, then $E[X] \le E[Y]$ with equality if and only if $X \le Y$ almost surely, then $E[X] \le E[Y]$ with equality if and only if $X \le Y$ almost surely. 1.3 The Measure Extension Theorem 31 μ is called the product measure or Bernoulli measure on Ω with weights (pe) $e \in E$. Then, the following are equivalent: $n \rightarrow \infty$ (i) There is a random vector X such that $Xn \Rightarrow X$. By Theorem 9.35, $Y := (Xn2)n \in I 2$ is a submartingale. $\theta + \rho \neq Exercise 2.2.2$ (Box-Muller method) Let U and V be independent random variables that are uniformly distributed on [0, 1]. (ii) There is an $f \in V$ with F(x) = x, $f \neq f$ and Y = 0, $f \neq f$ and Y = 0. If $L[X0] = \pi$, then X is stationary. any function $f: I \rightarrow R$ and any $n \in N$, define the nth lower sum and upper sum, respectively, by Ltn (f) := n n n (tin - ti - 1) inf [ti - 1, tin), i=1 Unt (f) := n n n (tin - ti - 1) sup f[ti - 1, tin), i=1 Unt (f) := n n n (tin - ti - 1) sup f[ti - 1, tin). Further, let Dn := sup $|Fn(t) - F^n(t)|$. used to provide familiarity with the techniques that will be needed for the more challenging classification of the infinitely divisible probability measures on R. (iii) Modify the argument in order to show that for $\alpha > 2$, the α -stable distributions in the broad sense are also necessarily trivial. The strong law of large numbers does not yield recurrence immediately and we have to do some work: By the Markov property, for every $N \in N$ and every y = x, GN(x, y) := N k=0 $\lambda \uparrow 1 [-\pi,\pi)D$ Re $1 1 - \lambda \phi(t) dt = \infty$. Show that XY δ has a continuous density. Let E be another Polish space. Ω1 Proof The proof follows the usual procedure of stepwise approximations, starting with an
indicator function., Xk) = PE ⊗k. Theorem 17.42 (Chung-Fuchs [27]) An irreducible random walk on ZD with characteristic function φ is recurrent if and only if, for every $\varepsilon > 0$, $\lim \lambda \uparrow 1$ ($-\varepsilon,\varepsilon$)D Re 1 1 - $\lambda \phi(t)$ dt = ∞ . In particular, Nt is a random variable; that is, measurable., E12 (but not E4) is a π -system., N - 1, and 18.4 Speed of Convergence 457 where $\rho = 2 r/(1 - r)$ and $\theta \in C \setminus \{-1, +1\}$ with $|\theta| = 1$. We call $C \subset Cb$ (E; C) a separating class for Mf 3(E) if for any 3 two measures $\mu, \nu \in Mf(E)$ with $\mu = \nu$, there is an $f \in C$ such that $f d\mu = f d\nu$. Using the strong Markov property of X (see Theorem 17.14), we get (''' (Px $\tau yk < \infty = Ex Px \tau yk < \infty = Fx + 11{\tau k-10}$, then y is also recurrent, and F (x, y) = F (y, x) = 1. For this distribution, we introduce the symbol $U\Omega := \mu$. The number $\rho\omega$ is called the weight of μ at point ω . By the discrete Fourier inversion formula (Theorem 15.11), ϕ is the characteristic function of the 3π probability measure $\mu \in M1$ (Z) with $\mu(\{x\}) = (2\pi)-1 - \pi \cos(tx) \phi(t) dt$. 20.1 Definitions Definition 20.1 Let $I \subset R$ be a set that is closed under addition (for us the important examples are I = N0, I = Z, I = R, $I = [0, \infty)$, I = Zd and so on). Furthermore, P0 [Yt1 is even] \approx 12. Now choose successively ri+1 \in Bci /2 (ri) and ci+1 \in (0, ci /2) such that qi+1 \in Bci +1 (ri+1). Lemma 14.21 The map f * g is measurable and we have f * g = g * f and (f * g) d\lambda = n Rn n Rn f d\lambda n Rn g d\lambda. Takeaways A family of measures is called tight if for larger and larger compacts, there is arbitrarily little mass outside the compact. μ is \emptyset -continuous. In words, we choose a random site $i \in \Lambda$ (uniformly on Λ) and invert the spin at that site. 23.2 Large Deviations Principle 597 Now let $U \subset R$ be open. If $A \in A$, then clearly also $1A \times E L1$ (P). (iii) If E is separable, then it can be shown that (Mf (E), τw) is metrizable; for example, by virtue of the socalled Prohorov metric. By construction, $-\log 2$ (pe) $\leq l(e) \leq 1 - \log 2$ (pe). By (ii) and Lemma 4.3, this implies ($\alpha f + \beta g$) $d\mu = \alpha \ln n \rightarrow \infty$ fn $d\mu + \beta g d\mu$. Proof Let An0 $\downarrow \emptyset$ be a decreasing sequence such that $|E \setminus An0| < \infty$ and $x1 \in An0$ n for all $n \in N$. be i.i.d. variables with E[Y1] = 0 and $Var[Y1] = \sigma 2 > 0. The additional statement holds since #C p (x) = y \in Zd 1 \{x \leftarrow \rightarrow p y\}$. Let (Rn)n \in N0 be an independent family of random variables with values in E E and with the property P[Rn (x) = y] = p(x, y) for all x, y \in E., α n such that $h = nk=1 \alpha k 1(tk-1, tk]$. #G & $g \in G 20.2$ Ergodic Theorems In this section, (Ω , A, P, τ) always denotes a measure-preserving dynamical system. m=n (ii) De Morgan's rule and the lower semicontinuity of P yield . \bigstar 56 2 Independent family of infinitely many events is given by the perpetuated independent repetition of a random experiment. (iii) Let $A \subset \Omega$. As en($\phi n - 1$) is the CFP of CPoinun , we infer CPoin $\mu n \rightarrow n \rightarrow \infty \mu$. • Due to thermic fluctuations, the state of the system is random and distributed according to the so-called Boltzmann distribution π on the state space $E := \{-1, 1\}\Lambda$. Then $P[X + Y = n] = e - (\mu + \lambda)n$. + Yn for $n \in N$. In the second section, we prove the convergence theorem. Then E[X1] = 0 and $\sigma 2 := Var[X1] = 1/(1 - 2\alpha) < \infty$ if $\alpha < 1/2$. 17.5 to prove recurrence are not very robust and would need a substantial improvement in order to cope with even a small change. A1 If κ is stochastic and if μ is a probability measure, then $\mu \otimes \kappa$ is a probability measure. For $\omega \in Ac$, we define X conditions, X is F-adapted. 2 Summing up, we have $|f| d\mu \leq \epsilon$. 17.6 Invariant Distributions. We make the ansatz $\lambda = (1 - r)\rho(\theta + \theta)$ and $xk = \rho k (\theta k - \theta k) 1 - r \lambda - 1 x1$ for k = 1, This is a crucial property that will be needed later., n} black balls is given by the hypergeometric distribution with parameters B, W, $n \in N$: B W b n-b HypB, W; n {b} = B + W n for b $\in \{0, ...\}$ Example 15.17 Define the function $\phi: R \rightarrow [-1, 1]$ for $t \in [-\pi, \pi)$ by $\phi(t) = 1 - 2|t|/\pi$, and assume ϕ is periodic (with period 2π). We abbreviate $Lx[Z^n] := Lnx!$) -1 (n Z nt !) $t \ge 0^*$. Proof By the Cauchy-Schwarz inequality, $|\phi(t) - \phi(s)|^2 = Rd 2$ ei) $t, x^* - ei)s, x^* \mu(dx) 2$ i) $t - s, x^* e = -1$ ei) $s, x^* \mu(dx) Rd 2$ i) $t - s, x^* i)s, x^* 2 e e \mu(dx) \le -1$ $\mu(dx)$ Rd = Rd Rd i)t $-s_x - 1 = -i$ t $-s_x -$ K. Proof As we need these statements only in the proof of the multidimensional Jensen inequality, which will not play a central role in the following, we only give references for the proofs. \bullet Remark 20.27 Clearly, (20.8) implies "ergodic". In particular, the case 1 = r1 = r2 = . The same computation with k = N - 1shows that (18.13) holds if and only if $\theta N - \theta N = 0$; that is, if $\theta 2N = 1.4$, $xn \in A$ such that $A \subset B\varepsilon(xi)$. }). Definition 1.8 A class of sets $A \subset 2\Omega$ is called a ring if the following three conditions hold: (i) $\emptyset \in A$. Hence, it is enough 3 to show that $f d\mu 1 = f d\mu 2$ for all $f \in Cc$ (Rd). We start by collecting some properties of the space $\Omega = C([0, \infty)) \subset I$ $R[0,\infty)$. x 1 2 x 1 0 2 0 2 Fig., $Bn+1 \in B(E)$). 5). (Compare Sect. By the transformation formula for densities (Theorem 1.101), the distribution of X has the density 1 1 x -1 exp - log(x) 2 f (x) = $\sqrt{2} 2\pi$ for x > 0. (vi) Consider an urn with $B \in N$ black balls and $W \in N$ white balls. aci = i=1 lim sup ε log(aci). Exercise 5.4.1 Let X1, The "space" in "space average" is the probability space in mathematical terminology, and in physics it is considered the space of admissible states with a certain energy (Greek: ergon). The aim is to simulate a Markov chain X with transition matrix p on a computer. On the other hand, for a function ϕ that is Hölder-y -continuous at a given point t, there need not exist an open neighborhood in which ϕ is continuous. + Xn n $\rightarrow \infty$ \Rightarrow N0,1. 6.1 Almost Sure and Measure Convergence In the following, (Ω , A, μ) will be a σ -finite measure space. Linearity and positivity are obvious, and the triangle inequality is a consequence of Minkowski's inequality, which we will show in Theorem 7.17. In the following, we will not need these statements. k=1 d $3d < \infty$, and so, by the Borel-Cantelli Therefore, $\infty k=1$ P[Mk-1 $\in \{0, L\} \le 2dL$ d lemma, $M \infty \in
\{0, L\} \le 2dL$ d lemma, M1} n→∞ and ' (P lim inf An \in {0, 1}. Proof For every n \in N, we have $\nu n \in$ M1 (R) since νn (R) = kn x 2 PXn, l (dx) = l=1 kn Var[Xn, l] = 1. Corollary 14.26 (Products via kernels) Let ($\Omega 1$, A1, μ) be a finite measure space, let ($\Omega 2$, A2) be a measurable space and let κ be a finite transition kernel from $\Omega 1$ to $\Omega 2$. As t $\rightarrow \alpha$ (t) is monotone increasing, this $-t 2 E[X2] + \varepsilon(t) t 2 2$ with $\varepsilon(t) \rightarrow 0$ for $t \rightarrow 0$. Proof If $\theta(p) = 0$, then by (2.14) $\psi(p) \leq P[\#C p(y) = \infty] = y \in Zd$ $\theta(p) = 0$. Proof Let (un) n \in N be a sequence in M ≤ 1 (E). ≤ 9.3 Discrete Stochastic Integral So far we have encountered a martingale as the process of partial sums of gains of a fair game. Hence, k 3 if we let $\nu = \infty$ to $(-1,1) \ge 2 \nu(dx) < \infty$. Example 8.18 Let X, Y \in L1 (P) be independent. Then X is a martingale if and only if $E[X\tau] = E[X0]$ for any bounded stopping time τ . Corollary 17.10 A stochastic process (Xn)n \in N0 . Lemma 15.45 If (i) of Theorem 15.45 holds, then (Xn,1) is a null array. 12.3 De Finetti's Theorem 269 Proof "=" This follows as in the proof of Theorem 12.24. Hence, it is not enough to consider pairs only. If i is the number of the urn from which the ball is drawn, then with probability p(i, j) move the ball to the urn with number j . , xn-1) and $0 \le k < n-1$, let $p(k, x) = \pi(\{xk\})P(xk)$. xk+1) · · · P (xn-2, xn-1). Proof We compute $x,y \in E$ (w(x) - w(y))I (x, y) = $w(x) w(y) I(x, y) - I(x, y) x \in E$ $y \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $y \in A$ $x \in E$ $x \in A$ $y \in E$ $x \in A$ $x \in E$ $x \in A$ continuous, symmetric, real function with $\phi_{\mu}(0) = 1$. 20.6 Entropy . bm, p = bm + n, p for m, $n \in N$ and $p \in [0, 1]$. Changing the roles of x and y in the above argument, we get F(x, y) = 1. For distributions on R, the problem is equivalent to finding a weak limit point for a sequence of distribution functions. (i) Show that Mn := Xn - n - 1 k = 0 d(Xk + 1) d() defines a martingale M with square n-1 variation process)M*n = i=0 f (Xi) for a unique function $f: E \rightarrow [0, \infty)$. Show that E[Xi | Sn] = 1 Sn n for every i = 1, The most prominent role is played by the Euclidean space Rn; however, we will also consider the (infinite-dimensional) space C([0, 1]) of continuous functions $[0, 1] \rightarrow R$. Then there exists a regular conditional distribution i $P[Z1 \in \cdot |Z1 + Z2 = x]$ for $x \in R$. Choose a parameter $p \in [0, 1]$ and an independent family of identically distributed random variables p p p p (Xe) $e \in E$ with Xe ~ Berp ; that is, P[Xe = 1] = 1 - P[Xe = 0] = p for any 74 2 Independence Fig. , N} and with the convention ml := #{r $\in \{1, ..., N\}$ and with the convention ml := {r $\{1, ..., N\}$ and w determined by its value on {1}, as $\mu = 0$ and $\nu = \delta 2$ are different finite measures that agree on E. be i.i.d. real random variables with E[X1] = 0 and E[X1 | k] < ∞ for all $k \in N$. be an arbitrary enumeration of O. \blacklozenge Remark 9.26 The etymology of the term martingale has not been resolved completely. If the test functions are also assumed to have compact support, we get vague convergence of measures. < tn and B0, . (2.10) $j \in J$ Corollary 2.23 Let $n \in N$ and let $\mu 1$, . Clearly, the sets (τn (E0), $n \in Z$) are disjoint and $E = n \in Z \tau n$ (E0). Hence, let I be a countable set and let (Bi)i $\in I$ be a countable set and l collect some simple properties of Hölder-continuous functions. The black dots are the Ones. Hence the fair price $\pi(VT)$ is determined uniquely once there is one trading strategy H and a v0 such that $VT = v0 + (H \cdot X)T$. By the measure extension theorem (Theorem 1.53), μ^{2} can be uniquely extended to a σ -finite measure on $A = \sigma$ (Z R). \blacklozenge Most Markov processes one encounters have the strong Markov property. 1 $\phi(t) := E[ei)t, X^*] = ei)t, \mu^* e - 2$)t, $Ct * for every t \in Rd$. In particular, and define A := An. By the Markov property, we have *) $\kappa t + s(x, A) = Px [Xt + s \in A] = Ex PXs [Xt \in A] = Px [Xs \in dy] Py [Xt \in A]$ $s(x, dy)s(y, A) = (ss \cdot st)(x, A)$. By the Riesz-Fréchet theorem (here Corollary 7.28), there exists a $g \in L2(\Omega, A, \mu + \nu)$ such that $h d\nu = hg d(\mu + \nu) = f d\mu$ for all $f \in L2(\Omega, A, \mu + \nu)$. Details can be found, e.g., in [37, Theorem 10.2.6]. 23.4). as in the proof of Lemma 13.5. Let N {x1, x2, . Hence, let $\mu = 0$. i=1 Evidently, compact sets are totally bounded. 1 n Rn n $\rightarrow \infty - \rightarrow 0$. Each of these four random variables is manifestly FT -1 -measurable. Applying this formalism we have been able to describe the phase transition of the Weiss
ferromagnet. It is called ergodic if there are no nontrivial

(w.r.t. P) invariant sets. Thus we can arrange the eigenvalues by decreasing modulus $\lambda 1 = 1 \ge |\lambda 2| \ge ... \omega n$ $) = \infty \mu(A^{\sim} k)$. In the case of monotone convergence we have equality. This contradicts the assumption and thus (i) holds. (ii) If (Mn)n \in N0 is a martingale with M0 = 0 and if there is a sequence (ck)k $\in N$ of nonnegative numbers with |Mn $n \in N$, define the map $fn : [0, 1] \rightarrow [0, 1]$ by $fn : x \rightarrow x n$. Takeaways A signed measure is a finite measure that can also assume negative values. \blacklozenge Let $T \in N$ be a fixed time. Assume further that there is an r > 0 such that supn $\in N$ Mr (Xn) $< \infty$. (i) If $E[\phi(Xt) + 1] < \infty$ for all $t \in I$, (9.1) then $(\phi(Xt) + 1] < \infty$ for all $t \in I$, (9.1) then $(\phi(Xt) + 1] < \infty$ for all $t \in I$, (9.1) then $(\phi(Xt) + 1] < \infty$ for all $t \in I$, (9.1) then $(\phi(Xt) + 1] < \infty$ for all $t \in I$, (9.1) then $(\phi(Xt) + 1] < \infty$ for all $t \in I$. sides. Hence $U = \infty$ i=1 Bi is an open set $U \supset A$ with λn ($U \setminus A$) < ε . Y). Then $f \circ X \in L1$ (μ) and ($f \circ X$) $d\mu = f d \mu \circ X - 1$. That is, $A \in \sigma$ ((Xe) $e \in E \setminus F$) for every finite $F \subset E$. \blacklozenge Theorem 24.12 For every $\mu \in M(E)$, there exists a Poisson point process X with intensity measure μ . i=1 Now assume the gambler adopts the following doubling strategy. P (T \leq t = π be a further, Proof Without loss of generality, assume T = 1 and $\zeta = \zeta 1$. (Ω , A*, μ *) is called the completion of (Ω , A, μ). Often α is also easy to obtain (e.g., via the representation from Exercise 16.1.3). For the proof of that theorem, only (17.5) was needed. Then construct Uf from such Uf δ , ϵ . (ii) For every N, we have lim sup V N (ω , δ) = 0. Thus there exists a C > 0 such that, for every $\mu \in M1$ (E), we have $\pi n \mu pn$ ({1, . Hint: Consider the random variable Y with respect to the probability measure Xm P[·]/E[Xm] = P[Y \in A] for all $A \in B(R)$ and $m \in N0$. Let $B = (Bt, t \in [0, 1])$ be a Brownian motion and let Xt := Bt - tB1. Definition 23.7 (Large deviations principle) Let I be a rate function and (µε) $\epsilon > 0$ be a family of probability measures on E. Hence (Xn) n \in N is exchangeable. Since conditional expectations are defined only up to equalities a.s., all equalities with conditional expectations are understood as equalities are defined only up to equality a.s., all equalities with conditional expectations are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined only up to equality a.s., all equalities are defined on a set of stopping times $\sigma K = \inf\{n \in N : |Xn| \ge K\}$; $\sigma K \pm = \inf\{n \in N : \pm Xn \ge K\}$ and τK as in the proof of Theorem 11.14. X and Y are called (i) modifications or versions of each other if, for any $t \in I$, we have Xt = Yt P-almost surely, \mathbb{C} The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. Let $B \subset \mathbb{C}$ E be closed and let $\varepsilon > 0$. $n \rightarrow \infty$ $n \rightarrow \infty$ 284 13 Convergence of Measures Proof "(iv) $\Rightarrow (v)$ " This is trivial. Let $\Lambda \subset \mathbb{Z}d$ be a set that we interpret as the sites at each of which there is one voter. Show that $\mu \varepsilon := \mathbb{N}0, \varepsilon^2$ satisfies an LDP with good rate function I (x) = $\infty \cdot 1\mathbb{R}\setminus\{0\}$ (x). i=1 Let An (ϕ) be the symmetrized average from Theorem 12.17. Proof Let kn = 2n and l(n) = n1/2 (log(n))(1/2)+ ε for $n \in N$. Furthermore, it is easy to see that X has Fig. x3 R3 x2 19.5 Network Reduction 483 Application to Example 19.32. Then $|f|p \neq 1 + |f|p$; hence $|f|p \neq \mu < \infty$. Example 5.24 (Shannon's theorem) Consider a source of information that sends a sequence of independent random symbols X1, X2, "(iii) \Rightarrow (i) \Rightarrow (ii) This is trivial. By the Markov property, for $x \in A$ and $y \in E$, * Ex g(X τ) X1 = y =) g(y), Ey [g(X τ)], if $y \in A$ if $y \in E \setminus A = f(y)$. Definition 7.47 (Dual space) Let (V, \cdot) be a Banach space. Unless otherwise stated, the vector spaces C(E) Cb (E) and Cc (E) are equipped with the supremum norm. The third section is devoted to applications of the convergence theorem to computer simulations with the so-called Monte Carlo method. If Ω is finite, then so is μ . (18.3) Proof (i) Let m, $n \in \mathbb{N}0$ with pm (x, y) > 0 and pn (y, z) > 0. Hence, in general, τ is not a stopping time. Exercise 21.8.1 Let $X_1, X_2, (i) \Rightarrow (i)$ The image measure PX describes the distribution of X. nx! (21.43) 556 21 Brownian Motion n Evidently, Ex [Z th] = nx! n $\leq x$ for every n; hence (Lx [Zt], n $\in N$) is tight. We consider the transition matrix r, if $j = i + 1 \pmod{N}$, p(i, j) = 1 - r, if $j = i - 1 \pmod{N}$, p(i, j) = 1 - r, if $j = 1 - 1 \pmod{N}$, p(i, j) = 1 - r, if j =0, else. If all Xi are real-valued, then the Cesàro limits 1 Xi n n lim inf $n \rightarrow \infty$ 1 Xi n n and lim sup $n \rightarrow \infty$ i=1 i=1 are also almost surely constant. (13.15) A Now let (X i) i \in I be a family of random variables on $[0, \infty)$ with E[Xi] = 1. 290 13 Convergence of Measures Let f be continuous and uniformly integrable with respect to (μ n) $n \in \mathbb{N}$ and assume 3 $n \rightarrow \infty$ that $\mu n \rightarrow \mu$ weakly. (i) If A is a ring, then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any two sets A, $B \in A$. If C is only positive semidefinite (and symmetric, of course), we define N μ , C as 1 that distribution on Rn with characteristic function $\phi(t) = ei)t$, $\mu^* e - 2$)t, Ct *. There are four elementary transformations for the reduction of an electrical network: 1. • Takeaways The speed at which a Markov chain converges towards its invariant distribution is determined by the spectral gap of its transition matrix. i=1 In other words, Z is exponentially distributed with parameter $\theta 1 + .$ For $f \in L2$ (µ1), define (Af)(t2) = a(t1, t2) f(t1) µ1 (dt1). "(i)" For $n \in N0$ and x, y, $z \in E$, by construction,) x * Px [Xn+1 = taking the spectral gap of its transition matrix. z Fn, $Xn = y] = P Xn + 1 = z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Xnx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Xnx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Xnx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Xnx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$, $m \leq n$, $Xnx = y^*$ = P Rn + 1 (Ynx) = $Z \sigma \text{ Rm}$ ($Ynx =
Y^*$) = P Rn + 1 ($Ynx = Y^*$) = P Rn + 1 (YTheorem 7.37 Let μ and ν be measures on (Ω , A). For A \subset E, we denote by A the closure of A, by A° the interior and by ∂A the boundary of A. Analogously, we write $f \geq 0$ and so on. 464 19 Markov Chains and Electrical Networks Further, define FA for pA similarly as F was defined for p. 24.2 Properties of the Poisson Point Process. Then n n $\rightarrow \infty$ PSn* $- \rightarrow N0, C$ weakly. That is, $P[X = n] = e - \lambda \lambda n / n!$ for $n \in N0$. Uniqueness of the Infinite Open Cluster* Fix a p such that θ (p) > 0. Since $G(x, y) = F(x, y) = Px[\tau^2 y < \infty] = F(x, y)$ for all $x \in E \setminus A$, $y \in A$. (4.3) 3 3 3 If we only have $f - d\mu < \infty$, then we also define f $d\mu$ by (4.3). & Exercise 14.4.5 Use the methods developed in this section to construct a stochastic process (Xt)t ≥ 0 with independent and stationary Poisson-distributed increments. n n Fn (x-) = i=1 Formally, define F ($-\infty$) = 0 and F (∞) = 1. If in (ii) we also have $\kappa(\omega 1, \Omega 2) \leq 1$ for any $\omega 1 \in \Omega 1$, then κ is called sub-Markov or substochastic. By an iteration procedure, show the even stronger statement D X(n), X(n-1), Assume that the computer provides a random number generator that generator is incident to a point $x \in TL$. We get things to work out better if we modify the definition: $Fn = [-n/2, (n+1)/2] \cap Z$. l=1 l=1 15.5 The Central Limit Theorem 359 In the following, ϕn , l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteristic functions of Xn, l and ϕn will always denote the characteri the geometry of the domain and is thus more universal than the Riemann integral. (i) f is locally Hölder-continuous of order $\gamma \in (0, \gamma)$. Since (Y - Y) = 0, we have P[A] = 0; hence $Y \leq Y$ almost surely. i=1 Let T⁻tK, n and U⁻ tK, n be the linearly interpolated versions of tK, n := $\sqrt{1}$ T Knt ! T σ 2n tK, n := $\sqrt{1}$ U Knt ! U σ 2n and for t ≥ 0 . 24 in a more general setting. Let X be a random point that is uniformly distributed on the surface. If only PXt = PX0 holds for every t \in I (without the independence), then in general, X is not stationary. Step 5. \blacklozenge Example 2.10 Let $\Lambda \in (0, \infty)$ and $0 \leq \lambda n \leq \Lambda$ for $n \in \mathbb{N}$. (ii) Every σ -ring is a random point that is uniformly distributed on the surface. If only PXt = PX0 holds for every t \in I (without the independence), then in general, X is not stationary. Chebyshev inequality holds:) * P $|X - E[X]| \ge \varepsilon \le \varepsilon - 2$ Var[X]. $E[\phi(X) \ En] = An(\phi) := n! (12.3) \in S(n)$ Proof Let $A \in En$. The map $\Gamma(s, t) := Cov[Xs, Xt]$ for s, $t \in I$ is called the covariance function of X. Here R1 = 5, R2 = 2, R3 = 56, $\delta = 95/6$, $R \ R2 = \delta/R2 = 95/12$ and $R3 = \delta/R3 = 19$. This will imply that ft is bounded also on the compact set [-1, -1, -1] = An(\phi) = n! (12.3) \in S(n) 1]. $n = -\infty$ This series converges in L2, and the sequence of square summable coefficients (cn) $n \in Z$ is unique (compare Exercise 7.3.1 with cn = (-i/2)an + (1/2)bn and c-n = (i/2)an + (i/2 Nx, we have *)*) $E \phi(X)|F(\omega) \ge \phi(x) + E D + \phi(x)(X - x)$ $F(\omega)(8.8) = \phi(x) + D + \phi(x)E[X|F](\omega) - x =: \psi\omega(x)$. We come back to this point in more detail in Chap. For an illustration of the inclusions between the classes of sets, see Fig. Now, for any $z \in [0, 1], 1 \ge \infty$ $\psi S n (z) k \to \infty = \psi S n - Skn(z) \ge 1 - P[S n - Skn \ge 1] \ge 1 - pn, 1 \to 1, \psi Skn(z)$ l=k+1 3.3 Branching Processes 91 hence ψ S n (z) = $lim \psi$ Skn (z) = $k \rightarrow \infty \infty$ (pn, l = t + 2 for every $t \in \mathbb{R}$. We only consider l=1 , Proof of Theorem 15.44 2 "(i) \Rightarrow (ii)" We have to show that $lim \log \phi_n(t) = -t2$ for every $t \in \mathbb{R}$. We only consider the case d = 1 (see [120, Thm. By construction, $\phi(x) + (y - x)t \le \phi(y)$ for all y < x if and only if $t \ge D - \phi(x)$. Definition 21.21 Let E be a Polish space. Let N be the smallest (random) nonnegative integer n such that $Un \le f(Xn)/c$ and define Y := XN. \blacklozenge 7.6 Supplement: Dual Spaces 187 We are interested in the case $V = Lp(\mu)$ for $p \in [1, \infty]$ and write F p for the norm of $F \in V$. 21.1 for a computer simulation of a Brownian motion. We write Ex for expectation with respect to Px, Lx [X] = Px and $Lx [X | F] = Px [X \in \cdot | F]$ (for a regular conditional distribution of X given F). Theorem 17.30 For all x, $y \in E$ and $k \in N$, we have) * Px $\forall y \in X$ and $Lx [X | F] = Px [X \in \cdot | F]$ (for a regular conditional distribution of X given F). Theorem 17.30 For all x, $y \in E$ and $k \in N$, we have) * Px $\forall y \in X$ and $Lx [X | F] = Px [X \in \cdot | F]$ (for a regular conditional distribution of X given F). this implies that $\lim \sup f - fnk \ 1 \le \lim \sup k \to \infty$ contradicting (6.6). Then $Sn - dn \ n \to \infty \Rightarrow \mu$. Then there exists a unique stochastic kernel κ from (E, B(E)) to (E I, B(E) \otimes I) with the property: For all $x \in E$ and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for any choice of finitely many numbers 0 = t0 < t1 < t2 < ., 6 and for an converges pointwise to a continuous function if and only if the limiting function is a characteristic function and the corresponding probability measures converge weakly., 6} and $\Omega 2 = \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 =
\omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = \omega = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$, then $\Omega 1 \times \Omega 2 = (\omega 1, \omega 2) : \omega 1 \in \{1, 2, 3\}$. $> -\infty$ almost surely and $\tau := \sup\{n \in N0 : Sn = S - \}$ is finite almost surely. This is obvious since $D - \phi$ and $D + \phi$ are the limits of the sequences of slopes of the left-sided and right-sided secant lines, respectively. Hence, for pairwise disjoint, measurable sets A1, . Hence we define the real random variable X as the sum of independent random variables, $X = b + XN + X0 + \infty$ ($Xk - \alpha k$), (16.8) k=1 where $b \in R$, $XN = N0, \sigma 2$ for some $\sigma 2 \ge 0$ and PXk = CPoivk with intensity measure νk that is concentrated on $Ik := (-1/k, -1/(k + 1)] \cup [1/(k + 3 1), 1/k)$ (with the convention $1/0 = \infty$), $k \in N0$. By the $n \rightarrow \infty$ Borel-Cantelli lemma, $\mu(B(\epsilon)) = 0$. We come back to this point in Theorem 5.36. The event is called n-symmetric if we allow only permutations of X1, Conclude that if F is the distribution function of a Stieltjes measure μ on R and if d F (x) = 0 for λ -almost all $x \in A$. Definition 13.26 (Tightness) A family $F \subset Mf(E)$ is called tight if, for any $\epsilon > 0$, there exists a compact set $K \subset E$ such that sup $\mu(E \setminus K) : \mu \in F < \epsilon$. $n \to \infty$ (ii) $E[f(Xn)] \to 0$ E[f(X)] for all $f \in Cb(R)$. We generalize this observation in the following theorem. (vii) $\mu = v$ -lim μ n and $\mu(E) = \lim \mu n(E)$. In fact, $\Omega 1 \times \Omega 2 \in G$ is trivial. 6.1 Almost Sure and Measure Convergence 151 Proof Clearly, condition (i) implies (ii) since Markov's inequality yields that $p \mu(\{|f - fn| > \epsilon\}) \le \epsilon - p f - fn p$., $XtnN \rightarrow Xt1$, . . , Xk-1) A E fk (X1) A. p q (7.1) x p y q + -xy for $x \in [0, \infty)$. However, by Birkhoff's ergodic theorem, 1 n If we define A := Thus P $\perp Q$. Proof The proof is simple and is left as an exercise. In particular, $X \sim exp\theta$ if and only if $P[X > t + s | X > s] = e -\theta t$ for all $s, t \ge 0$. Show that $\mu(K) < \infty$ for any compact set K. Hence also $A \cap B = A \setminus (A \setminus B) \in A$. As a simple application of Fubini's theorem, we can give a new definition for the convolution of, more generally, finite measures on Rn. In particular, the composition $\kappa_1 \cdot \kappa_2$ is a (sub)stochastic kernel from (Ω_0 , A_0) to (Ω_2 , A_2). Draw $n \in N$ balls from the urn without replacement. Theorem 14.22 (Convolution of n-dimensional measures) (i) If X and Y are independent Rn valued random variables with densities fX and fY, then X + Y has density fX * fY. Note that X1 \circ i is constantly ω^{\sim} 1 (and hence A1 -measurable), and X2 \circ i = id Ω 2 (and hence A2 -measurable). If in addition, $\phi(tn) = 1$ for all n, then X = 0 almost surely., Yk are independent with distribution Ξ^{∞} . (A somewhat more systematic proof is based on the factors of the that (f, g) is A - B(E × E)-measurable (this will follow from Theorem 14.8) and that d : E × E → [0, ∞) is continuous and hence B(E × E) - B([0, ∞))-measurable. Since E is Polish, E N0 is also Polish and we have B(E N0) = B(E) \otimes N0 (see Theorem 14.8). 82 2 Independence For three distinct points x 1, x 2, x 3 ∈ BL \ BL-1, let Fx 1, x 2, x 3 be the event where for any i = 1, 2, 3, there exists an infinite self-intersection free open path πx i starting at x i that uses only edges in E p \ EL and that avoids the points x j, j = i. A graph with vertex set (or set of nodes) E and with edge set K (see page 73). 21.5 Construction via L2 - Approximation 535 Exercise 21.4.1 (Doob's inequality) Let $X = (Xt) t \ge 0$ be a martingale or a nonnegative submartingale with RCLL paths. Using a Borel-Cantelli argument, it is not hard to show that this is exactly the conditional expectations as the defining equation for a fair game that in the following will be called a martingale. Then Ka := infx a H x(x) $\uparrow \infty$ if a $\uparrow \infty$. Finally, we define random variables as measurable maps. I (x1) 1=0 By symmetry, we also have Reff ($b \leftrightarrow n$) = n-1 R(1, 1 + 1) 1=k and thus Reff ($b \leftrightarrow n$) = Reff ($b \leftrightarrow n$). Define the free energy (or Helmholtz potential) per particle as $F \beta(x) := U(x) - \beta - 1 H(x)$. For $B \in F\sigma$, using the Markov property (in the third line), we get ∞)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \circ \tau n \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \in A$ n=0 $x \in E = \infty$)* $P\pi X \in B$, $\sigma = n$, Xn = x, $X \in A$ n=0 $x \in E = \infty$ whole family of distributions with the same moments as X. We now define N = (Nt)t $\in [0, 1]$ by Nt := L 1(0,t](X1) for t $\in [0, 1]$. Let $\Lambda(t) = \log E e$)t, X1 * for t $\in \mathbb{R}d$ (which is finite since Σ is finite) and $\Lambda * (x) = \sup E e$ (X1) to t $\in [0, 1]$. Let $\Lambda(t) = \log E e$)t, X1 * for t $\in \mathbb{R}d$ (which is finite since Σ is finite) and $\Lambda * (x) = \sup E e$ (X1) to t $\in [0, 1]$. Let $\Lambda(t) = \log E e$)t, X1 * for t $\in \mathbb{R}d$ (which is finite since Σ is finite) and $\Lambda * (x) = \sup E e^{-1} E e^{-1$ for every $t \in R$, -1/2) * E et X = $2\pi\sigma 2 = e\mu t + t 2\sigma 2/2 =$ $log(\pi(\{e\}))$. (q) = lim Fnk (q) and Define F k $\rightarrow \infty$ (q) : q \in Q with q > x. By Definition 17.3, a Markov process X = (Xn) n \in Now that p is reversible with respect to π if and only if the linear map L2 (π) \rightarrow L2 (π), f \rightarrow pf is self-adjoint. The complement of [$\omega 1$, σ = 1.5 models a state space). Exercise 19.2.1 Show that p is reversible with respect to π if and only if the linear map L2 (π) \rightarrow L2
(π), f \rightarrow pf is self-adjoint. The complement of [$\omega 1$, σ = 1.5 models a state space). Exercise 19.2.1 Show that p is reversible with respect to π if and only if the linear map L2 (π) \rightarrow L2 (π), f \rightarrow pf is self-adjoint. The complement of [$\omega 1$, σ = 1.5 models a state space). $+ \alpha N \mu N$, we have $\mu pn = N i = 1 \lambda i \alpha i \mu i \rightarrow \alpha 1 \pi$. 396 17 Markov Chains Theorem 17.11 Let I = N0. Denote by Px that measure on C([0, ∞)) for which X 0 = 0). Let $\xi_{0,1}$, $(\xi_{n,k})_{n \in \mathbb{N}}$, k=1,...,2n-1 be independent and N0,1 -distributed. n n=1 Hence F is tight. As a supplement, we cite a statement about the speed of convergence in the central limit theorem (see, e.g., [155, Chapter III, §11] for a proof). Radon measures are inner regular Borel measures (locally finite measures). ♦ Example 12.23 There is no "monotonicity" for conditional independence in the following sense: If F1, F2 and F3 are σ -algebras with F1 C F2 C F3 and such 12.3 De Finetti's Theorem 267 that (Ai)i ∈I is independent given F1 as well as given F3, then this does not imply independence given F2. Clearly, the set function α on C is monotone, additive and subadditive: $\alpha(C1) \leq \alpha(C2)$, $\alpha(C1 \cup C2) \leq \alpha(C1) + \alpha(C2)$, if $C1 \subset C2$, if $C1 \cap C2 = \emptyset$, (13.13) $\alpha(C1 \cup C2) \leq \alpha(C1) + \alpha(C2)$. As E is locally compact and separable, E is σ -compact. Now, G is a Dynkin system that contains a \cap -stable generator of A1 \otimes A2 (namely, the cylinder sets A1 \times A2 , A1 \in A1 , A2 \in A2). 3d Show by a counterexample that the condition of similarity of the open sets in U is essential. n $\rightarrow\infty$ Proof "(i) \Rightarrow (ii)" This follows by the Portemanteau theorem. (iii) Assume that, more generally, X is only adapted and integrable. Show that X is a martingale that converges almost surely., d-1 and Lx, y = -Ly, x (mod d). Now we check that A and B indeed fulfill (2.1). Rw (19.19) If X is transient, in which direction does it get lost? If $Ft = \sigma$ (Xs, $s \le t$) for all $t \in I$, then we denote by $F = \sigma$ (X) the filtration that is generated by X. The following theorem says that the converse also holds; that is, X is a martingale if, for sufficiently many predictable processes, the stochastic integral is a martingale. Define gn := inf fm . As an application we construct a certain subordinator and show that the Poisson point process is the invariant measure of systems of independent random walks. 23.3 Shifted free energy F β (m) – F β (0) of the Weiss ferromagnet with exterior field h = 0.04. 13.1 A Topology Primer . 7.2 Inequalities and the Fischer-Riesz Theorem . $n \rightarrow \infty$ i=1 "(iii) \Rightarrow (i)" Assume now that (iii) holds. , Yn are independent and Berx -distributed. Let E be a locally compact Polish space and let C0 (E) be the set of (bounded) continuous functions that vanish at infinity. Letter A B C D E F G H I J K L M Morse code .-..., xi-1, yi, xi+1, A different derivation, in contrast to the appearance, is based on the function of the rein, which is to "check the upward movement of the horse's head". In order to prove $(0 \leftrightarrow \infty) < \infty$. Exercise 15.3.1 (Compare [50] and [4]) Show that there exist two exchangeable sequences $X = (Xn)n \in N$ and $Y = (Yn)n \in N$ an $h \in N$ of real random variables with PX = PY but such that n k=1 D Xk = n Yk for all $n \in N$., n. All eigenvalues $\lambda 1$, .; ; lim; Xn - X; $\infty = 0$ P-almost surely. By Thomson's 19.4 Recurrence and Transience 473 principle, the principle of conservation of energy and the assumption R(x, y) $\leq R$ (x, y) for all x, $y \in E$, we have u(1) - u(0) = u(1) - u(0) I (A1) = I(x, y) 2 R(x, y) 2 R(E[Y Ft] for all $t \in I$. I Hence $h = g \lambda$ -a.e. By construction, $g \le f \le h$, and as limits of simple functions, g and h are B(I) - B(R)-measurable. As I and u are uniquely determined by x0, x1 and C, the quantities Ceff (x0 \leftrightarrow x1) are well-defined and can be computed from C. & Exercise 13.2.7 We can extend the notions of weak convergence and vague convergence to signed measures; that is, to differences $\phi := \mu + - \mu - \phi$ measures from Mf (E) and M(E), respectively, by repeating the words of Definition 13.12 for these classes. Hence the random variables X1, X2, ..., n}: ir = 1 for l \in \{1, ..., n\} ir = l for l $\in \{1, ..., n\}$ ir = l for l $\in \{1, ..., n\}$ ir = l for l $\in \{1, ..., n\}$ ir = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) ≥ 2 = l for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) = l for l = 0 for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) = l for l = 0 for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) = 0 for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn $\leq P$ Xn (k) = 0 for l $\in \{1, ..., n\}$ if $n \to \infty$ P Nt = Ntn (k) for l $\in \{1, ...,$ $2n P [N2-n t \ge 2] \rightarrow 0$. $i=1 i=1 \otimes N$. Now consider the general case where G is not necessarily open. Markov processes can be characterised by their transition probabilities (stochastic kernels). k=0 Theorem 20.16 (Lp -ergodic theorem, von Neumann (1931)) Let (Ω , A, P, τ) be a measure-preserving dynamical system, $p \ge 1$, $X0 \in Lp$ (P) and Xn = X0 $\circ \tau n$. Further, let E := ∞ En = σF : F : E N \rightarrow R is measurable and symmetric n=1 -1 and let En := ∞ n=1 En = X (E) be the σ -algebra. Here C(x, y) = 1{|x-y|=1}. It is easy to check that (bn,k) is a basis (exercise!). Reflection Find an example that shows that condition (iii) in Theorem 6.27 cannot simply be dropped. Then the limit (inferior) of the integrals is at least as large as the integral of the limit (Xi) i \in It) k \in K is independent. We thus get p2n (0, 0) = (2D)-2n k1 +...+kD = n where N 11 ,..., lr = N! 11 !… Ir ! 2n, k1, k1, . Proof Let (xn) $\in N$ be a sequence in E with lim xn = x0. i=1 Then β is a countable base of the/ topology τ ; hence the intersection AI := A $\subset \Omega$: A \in Ai for every i \in I = Ai $\in I$ is a σ -algebra. Clearly, p is the transition matrix of an aperiodic recurrent random walk on Z. (13.8) Let $\varepsilon > 0$. \$ 5.2 Weak Law of Large Numbers Theorem 5.11 (Markov inequality, Chebyshev inequality) Let X be a real random variable and let $f : [0, \infty) \rightarrow [0, \infty)$ be monotone increasing. Define the intervals $I0 = [1, \infty)$ and Ik = [1/(k + 1), 1/k) for $k \in \mathbb{N}$. That is, we define the Cramér transform $\mu^{2} \in \mathbb{M}$ (R) of μ by $\mu(dx)^{2} = -1$ $e_{\tau} x \mu(dx)$ for $x \in R$. This definition is parallel to that of a signed measure that is the difference of two finite measures. = ki ! = i=1 We close this section by presenting a further, rather elementary and instructive construction of the Poisson process based on specifying the waiting times between the clicks of the Geiger counter, or, more formally between the points of discontinuity of the map t \rightarrow Nt (ω)., d. One has to check that the intervals (a, bc] and so on can be chosen such that $\mu((a, bc]) + \varepsilon$. n=1 1An Summarising, D is a λ -system that contains a π -system that contains a π -system that contains a π -system that $\mu((a, bc]) + \varepsilon$. n=1 1An Summarising, D is a λ -system that contains a π -system that $\mu((a, bc)) + \varepsilon$. 13. Hence, by assumption, t] = E[X lim Q+ $s \downarrow t$, s > t E[Xs] = E[Xt]. Exercise 7.3.1 (Fourier series) For $n \in N0$, define Sn, Cn : [0, 1] $\rightarrow [0, 1]$ by Sn (x) = sin($2\pi n x$), Cn (x) = cos(2π Convergence Theorem Our goal is to use a coupling of two discrete Markov chains. In fact, $t \rightarrow
N^{\tilde{c}}$ t is not monotone. Formally, however, we can also define independence of the σ -algebras they generate. Definition 8.13 If Y is a random variable and $X \in E[X | Y] := E[X | \sigma(Y)]$. are also identically distributed, then the probability generating function of S is given by $\psi S(z) = \psi T(\psi X1(z))$. Furthermore, it shows that two ergodic measures are either equal or mutually singular. $\in L1(\mu)$. On the other hand, if Gn (x) = F (x - n), then (Gn)n \in N converges pointwise to $G \equiv 0$. Exercise 14.2.1 Show the following convolution formulas. Hence one can choose the constant sequence En = Ω , $n \in N$. A second possibility to spoil (iii) is to define N t := supr 0 und N 0 = 0., Xtn)-1 = $\delta x \otimes n-1 \kappa ti+1$ -ti ., Xn); hence $n \sigma (Y) \subset \sigma (X)$, and thus also $F = \sigma (X)$. For example, we could buy resistors in an electronic market, solder the network and measure the resistances with a multimeter. (i) If ϕ is differentiable at 0, then ϕ (0) = i m for some m \in R. For n \in N, define An, N := B1/n (xi). In particular, in this case, Mf (E) \subset M(E). The elements A \in B(Ω, τ) are called Borel sets or Borel measurable sets. Zn decomposes Fn subset Zn \subset Fn \ { \emptyset } such that $B = C \in Zn \ C \subset B$ into its "atoms"., $6\} \times A^2 \times \{1, . Let \ u1 = (1, 0, . \in E \text{ with } Nn \ \infty \ E = B1/n \ (xin).$ If we choose the orthonormal basis cleverly, then we automatically get a continuous with respect to μ . A Reflection Consider the case P[Y1 = 1] = P[Y1 = -1] = 12. Let $Sn * 1 := \sqrt{n\sigma 2 n}$ (Xk - μ k=1 be the normalized nth partial sum. Consequently, we have P[lim inf Sn = $-\infty$] = 1. \bullet (10.2) i=1 Definition 10.3 Let (Xn)n \in I be a square integrable F-martingale. It remains to close the gaps between the points {0, N, 2N, . $\epsilon \downarrow 0$ This implies (P5). For any measurable A \subset E, {Xi \in A} occurs for exactly NEN (A) of the i \in {1, . Hence E[X0] \leq E[XT The product-σ -algebra A = Ai i ∈ I is the smallest σ -algebra on Ω such that for every i ∈ I . the coordinate map Xi is measurable with respect to A - Ai ; that is, A = σ Xi , i ∈ I := σ Xi - 1 (Ai), i ∈ I . Evidently, the set A := y ∈ R : µ f -1 ({y}) > 0 of atoms of the finite measure µ \circ f -1 is at most countable. Then pm+n (x, z) \geq pm (x, y) pn (y, z) > 0. 53 53 61 69 73 3 Generating Functions . Joining serial edges. By the Borel-Cantelli lemma, there exists an N = such that N-1N(ω) ∞ |Xn| \leq K; hence Xn = Yn for all n \geq N. Why is this kind of limit incompatible with the concept of the Lebesgue integral? Thus there exists an open set D1 with B1 \subset D1 \subset D1 \subset D1 \subset D1 \subset D1 \subset D1 \subset A1. A \in Pn Similarly as in the simple shift case, we obtain the subadditivity of (hn) and thus the existence of h(P, τ ; P) = lim n $\rightarrow \infty$ 1 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim n $\rightarrow \infty$ 1 hn (P, τ ; P) = lim 5.5.) & Chapter 15 Characteristic Functions and the Central Limit Theorem The main goal of this chapter is the central limit theorem (CLT) for sums of independent arrays of random variables (Lindeberg-Feller theorem, Theorem 15.44). Klenke, Probability Theory, Universitext, 461 462 19 Markov Chains and Electrical Networks 19.1 Harmonic Functions In this chapter, E is always a countable set and X is a discrete Markov chain on E with For t \in D, define X sn \rightarrow t, the sequence Xsn (ω) n \in N is a Cauchy sequence. For D \geq 3, the sum over the multinomial coefficients cannot be computed in a satisfactory way. For stopping times, however, only retrospection is allowed. Then, for any $A \in E$ there exists a $B \in T$ with $P[A \ B] = 0$. Then lim inf fn dµ ≤ 1 im i=1 Takeaways For finite products of measurable spaces, we define the product measure. Therefore, A is sometimes called the increasing process of Y. We further assume that the U x,y,n are independent of X and Y. By the above, for any $n \in N$, we can choose an $Nn \in N$ such c n that $\mu(An, N) < \epsilon/2$ for all $\mu \in F$. (To be precise, we have shown only for $/n1/\alpha \ \alpha \in (0, 1]$ (in Corollary 15.26) and for $\alpha = 2$ (normal distribution) that $\phi \alpha, \gamma$ is in fact a CFP. As X is a supermartingale, for every $A \subset E$, $*P \ \xi n \ (X) = \nu P[\xi n \$ inf Iµ (A). Let $\phi: R \rightarrow [0, \infty)$ have the properties that • $\phi(ak) = yk$ for all k = 0, ., Xn] = g E[X1], . 21.9 Pathwise Convergence of Branching Processes 557 Lemma 21.47 The first k moments of Yt can be computed by differentiating the Laplace transform, Ex [Ytk] = $(-1)k dk x \psi(\lambda)$ Gaussian process. A probability measure μ on (E1 × E2 , E1 \otimes E2) with $\mu(\cdot \times E2) = \mu 1$ and $\mu(E1 \times \cdot) = \mu 2$ is called ergodic if (Ω , A, P, τ) is called a coupling of $\mu 1$ and $\mu(E1 \times \cdot) = \mu 2$ is called ergodic if (Ω , A, P, τ) is ergodic. We assume that (Ft)t \in I = F = σ (X) is the filtration generated by X. Let P be a finite measurable partition of Ω ; that is, P = {A1, . + Tn-1 and define X by Xt := sup n \in N : Sn \leq t for all t \geq 0. Example 16.4 For every measurable set A \subset R \ {0} and every r > 0, r -1 CPoirv (A) = $e - rv(R)v(A) + e - rv(R) \propto k-1 *k rv(A) k=2 k! r \downarrow 0 \rightarrow v(A)$ Further, let $E = \{0, . In the case of bond percolation on Z2, however, the exact value of pc can be determined due to the self-duality of the planar graph (Z2, E). Example 5.9 (i) Let <math>p \in [0, 1]$ and $X \sim Berp$. It is substochastic if $Ki \leq 1$ for all $i \in \Omega 1$. X is a submartingale if and only if A is monotone increasing. In particular, for countable time sets, the strong Markov property follows from the Markov property. If $\mu = P$ is a probability measure, then we say that E holds P -almost surely (a.s.), respectively almost surely (a.s.), respectively (a.s.), resp 18.8, there is a successful coupling for all initial states. × $\{en-1\} \times E \{n,n+1,...\}$. Since we have $n \ yk \ y^{-1}f(tk - tl) \ge 0$, k,l=1 in other words, if the matrix (f(tk - tl))k,l=1,...,n is positive semidefinite. Then $1 \ E[X] = \sqrt{2\pi\sigma} \ 2 \ model{eq:all-coupling} = \sqrt{2\pi\sigma} \ 2 \ mod$ $2\pi\sigma 2 x e^{-(x-\mu)} 2/(2\sigma 2) (x + \mu) e^{-x} \propto -\infty x e^{-x} dx 2/(2\sigma 2) 2/(2\sigma 2) dx (5.4) dx = \mu$. Exercise 23.2.6 Let X $\lambda \sim Poi\lambda$ for every $\lambda > 0$. (13.9) Proof "(i) = (i)" By the simple implication in Prohorov's theorem (Theorem 13.29(ii)), weak
convergence implies tightness., $\tau(d) < \infty$. 1/3. Definition 13.1 A topological space (E, τ) is called a Polish space if it is separable and if there exists a complete metric that induces the topology τ . Second Exercise 11.2.8 Let $p \in [0, 1]$ and let $X = (Xn)n \in N$ be a stochastic process with values in [0, 1]. Hence there is an $N \in N$ with $P[\tau < N] \ge 12$. The σ -algebra (2E) $\otimes N := \sigma$ (A) is called the We write $e \in E$ pe δe product σ -algebra on Ω . Hint: Use (ii) and the law of large numbers. Hence we make the following definition. 1 + 2 - Sn We conclude that (Zn) $n \in N0$:= (|Sn |) $n \in N0$ is a Markov chain on N0 with transition matrix [z 2/(1 + 2z), | | 1, p(z, z) = z | 1/(1 + 2), | | 1, p(z, z) = z + 1 > 1, if z = z +on Brownian motion (21) ix x Preface to the First Edition makes reference to Chaps. n > x Proof For $m \in N$, by comparison with the corresponding integral, we get ∞ $n-2 \le m-2 + \infty t - 2$ dt = $m-2 + m-1 \le m = m - 2 + m - 1 \le m = m - 2 + m - 1 \le m - 2 + m - 1 \le m = m - 2 + m - 1 \le m = m - 2 + m - 1 \le m - 2 + m - 2 + m - 1 \le m - 2 + m - 2 + m - 1 \le m - 2 + m - 2 + m - 1 \le m - 2 + m$ strategy itself is the second meaning of la martingale. \bullet Remark 12.8 If we write $\Xi n (\omega) := \xi n (X(\omega)) = n1$ ni=1 $\delta Xi (\omega)$ for the nth empirical distribution $\Xi \infty$. Since C = C0 + iC0, C is dense in Cb (E; C). Exercise 7.6.1 Show that $Ef \subset Lp(\mu)$ is dense if p $\in [1, \infty)$. Hence I (x) $\geq J$ (x). Hence E[X2] = E[X(X - 1)] + E[X] = n2 p2 + np(1 - p) and thus Var[X] = np(1 - p). Furthermore, $\kappa 1 \otimes \kappa 2 (\omega 0, An \times \Omega 2) \leq n \cdot \kappa 1 (\omega 0, An) < \infty$. Furthermore, $\kappa 1 \otimes \kappa 2 (\omega 0, An \times \Omega 2) \leq n \cdot \kappa 1 (\omega 0, An) < \infty$. I). Otherwise the bet is lost (for the player, not for the casino). Let $V := Q \cap I \circ A$ irreducible and aperiodic positive recurrent Markov chain is mixing. In the following, an important role is played by the function *) U(x) := E X2 1{|X| \le x}. Then F1 \subset F2 \subset . Define either B = C or B $\cap C = \emptyset$ holds. Thus the notion of a martingale might first have been used for general gambling strategies (checking the movements of chance) and later for the doubling strategy in particular, $m \in N(x, y)$, $n \in N(y, x)$ and $k \ge ny$, then kdy $\in N(y, y)$; hence $m + kdy \in N(x, y)$ and $m + n + kdy \in N(x, x)$. $\phi(X)$ En = E | n! n! \in S(n) \in S(n) Heuristic for the Structure of Exchangeable family X1, Define S n := ∞ l=1 Xn,l and Skn := k Xn,l for k \in N. (20.4) Indeed, we have {X \in A} = {X $\in \tau - n$ (A)} = {(Xn, Xn+1, .) For any A \in A], X]-1 (A) $\subset \Omega$ is called a cylinder set with base [.19,15] Simple ladder graph a z Fig (8.6) Lemma 8.10 The map E[X |F] has the following properties. 3.1 Definition and Examples 87 Example 3.4 (i) Let X be bn, p -distributed for some $n \in N$ and $p \in [0, 1]$. Since we have a prefix code, the sets CL (e) $\subset \{0, 1\}L$. 118 5 Moments and Laws of Large Numbers (ii) Let $n \in N$ and $p \in [0, 1]$. For ω $\in \Omega \setminus N$, define $F^{\sim}(z, \omega) := \inf F(r, \omega) : r \in Q, r > z$ for all $z \in \mathbb{R}$. (iv) Let r > 0 (note that r need not be an integer) and let $p \in (0, 1]$. If $(\Omega, A) = \mathbb{R}$, $B(\mathbb{R})$, then X is called a real random variable or simply a random variable. Clearly, $A2L \uparrow \{\mathbb{N} \geq 2\}$ for $L \to \infty$. Let $r := \mathbb{P}p[0 \in T]$. \blacklozenge Definition 1.106 If the distribution function $F : \mathbb{R}n \to [0, 1]$ is of the form $F(x) = x1 - \infty dt1 \cdots xn - \infty dtn f(t1, 21.5 Construction via L2 - Approximation 541 As shown above, the sequence (Xn) converges in L2 ([0, 1]) towards a process X, which (up to continuity of paths) has all properties of Brownian motion: Xt = <math>\xi 0 t + \infty n = 1 \sqrt{2} \xi n \sin(n\pi t)$. On the other hand, (16.4) implies $\phi n, n \to \infty$ every sequence (ln) with $\ln \leq \ln nx, y := nx + d d$ Owing to (18.4), we have $(nx, y + ny, z)d + Lx, y + Ly, z \in N(x, z)$. We say that E holds μ -almost everywhere (a.e.) or for almost all (a.a.) ω if there exists a null set N such that $E(\omega)$ holds for every $\omega \in \Omega \setminus N$. For periodic chains, the state space decomposes into d subspaces that can be entered at specific times (mod d) only • Next we show Jensen's inequality for conditional expectations. Remark 8.26 It is sufficient to check property (i) in Definition 8.25 for sets A2 from a π -system E that generates A2 and that either contains Ω^2 or a sequence En $\uparrow \Omega^2$. Finally, we consider the general situation. To this end, the voter chooses a neighbor In + Nn $\in \Lambda$ (with periodic boundary conditions; that is, with addition modulo L in each coordinate) at random and adopts his or her opinion. Now assume that all of the composed maps Xi \circ Y are A - Ai -measurable. Let (E, d) be a metric space. By construction, we have $g \le h$ and $g d\lambda = \lim I n \rightarrow \infty I$ and $d\lambda = \lim I n \rightarrow \infty I$ hn $d\lambda = h d\lambda$. On the same that all of the composed maps Xi \circ Y are A - Ai -measurable. Let (E, d) be a metric space. By construction, we have $g \le h$ and $g d\lambda = \lim I n \rightarrow \infty I$ and $d\lambda = \lim I n \rightarrow \infty I$ and $d\lambda = \lim I n \rightarrow \infty I$. other hand, by construction, we have qk \in i=1 BEi (ri) for all k \in N. Using the notation of variance and covariance, a simple proof looks like this: Case 1: Var[Y] = 0. We can represent the codes of all letters in a tree. By the local central limit theorem (see, e.g., [20, pages 224ff] for a one-dimensional version of that theorem or Exercise 17.5.1 for an analytic derivation), we have $n \rightarrow \infty$ nD/2 p2n (0, 0) = nD/2 P[Sn = 0] $\rightarrow 2$ (4 π /D)-D/2. Exercise: Prove the statements made above. A nice survey on MCMC methods including coupling from the past is [66]. Let 1, if the nth ball is black, Xn := 0, else, 270 12 Backwards Martingales and Exchangeability and let Sn = n i=1 Xi. We saw that almost sure convergence implies convergence in measure/probability. 3 3 Proof (i) Case Mf (E). As a consequence, the terminal σ -algebra and the exchangeable σ -algebra and the exchangeable σ -algebra coincide (mod P). Let N \in A with $\mu(N) = 0$. Theorem 4.10 (Image measure) Let (Ω , A) and (Ω , A) be measurable spaces, let μ be a measure on (Ω , A) and let X : $\Omega \rightarrow \Omega$ be measurable. We can define the conditional expectation as the monotone limit E[X | F] := lim E[Xn | F], $n \rightarrow \infty$ where $-X - \leq X1$ and $Xn \uparrow X$. Definition 1.48 ($\mu *$ -measurable sets) Let $\mu *$ be an outer measure. Takeaways A Polish space is a separable topological space that allows for a complete metric, e.g., the euclidian space Rd . For $\lambda \geq 0$, define the continuous function $f\lambda$ $[0, \infty] \rightarrow [0, 1]$ by fA (x) = $e - \lambda x$ if x < ∞ and fA (∞) = limx $\rightarrow \infty e - \lambda x$. We only sketch the argument., yn-1 + An k=1 μk (Ak) μn (An). (ii) This is a direct consequence of (i) since SA (x) $\supset E \setminus A$ for any $x \in E \setminus A$., Yn) given {X = x}'' should be (Berx) \otimes n. \clubsuit Exercise 5.3.7 Let m $\in (0, \infty)$ and let Wm = p = (pk) k $\in \mathbb{N}^{0}$ is a probability measure on N0 and ∞ kpk = m k=0 be the set of probability measures on N0 with expectation m. 413.2 Weak and Vague Convergence In Theorem 7.17, we verified the triangle inequality and hence that \cdot p is a norm. Replace the parallel edges by edges with resistances (12 = 1995 + 5) 10 and 6 19 -1 (19 + 1) = 25, respectively (right in Fig. The following are equivalent. Hence, by (21.7), $|Xt(\omega) - Xt|m \le (\omega) 2 - \gamma l \le l-1 l = n l = n 2 - \gamma n$. 1 19.5 Network Reduction 481 R (x, 1) x R (0, 1) x) 1 R (0, 1) 0 Fig. Macroscopically, this is the quantity that can be measured. If simple random walk on (E, K) is
recurrent, then so is simple random walk on (E, K) is recurrent. fg dµ. Clearly, we have $E = \sigma$ (A). Hence, by the π -A theorem (Theorem 1.19), $D = A1 \otimes A2$. Proof Define Y $n = (B2-n + t - B2-n) t \in [0, 2-n]$, $n \in N$. "(i) \Rightarrow (iii)" Let µ be a premeasure and $A \in A$. Hence one cannot infer that (X1, . If B is a Brownian motion) * (on some probability space (Ω, A, P)), then there exists an $\Omega \in A$ with $P \Omega = 1$ and $B(\omega) \in A$. $C([0, \infty))$ for every $\omega \in \Omega$. In the first category, we have characteristic functions, Laplace transforms and probability generating functions. $-1 + 0 \blacklozenge (5.5)$ Theorem 5.10 (Blackwell-Girshick) Let T, X1, X2, . (vii) Let Ω be an arbitrary nonempty set. For the latter, we prove only that one of the two implications (Lindeberg's theorem) that is of interest in the applications. If X1, X2, At the second stage, depending on the value of X, the values of Y = (Y1, . nm) such that $[\omega 1, .mn]$ suc $P[|Xn| > K] < \infty$. On the other hand, there exist weakly mixing systems that are not strongly mixing (see [81]). Similarly, for closed $C \subset Rd$, we have lim sup $n \to \infty 1$ log PSn /n (C) $\leq -\inf I^{\sim}(C)$. 19.8 Reduced network with three nodes. $\in A$ such that (a, b] $\subset \infty n=1$ (a(n), b(n)] and a < b. We will not go into the 17.1 Definitions and Construction 393 details but will henceforth assume that all Markov processes are time-homogeneous. & Exercise 6.1.2 Give an example of a sequence that (i) converges in L1 but not almost everywhere, (ii) converges almost everywhere but not in L1. real random variable X is characterized by its moments. $n \rightarrow \infty$ (i) fn $\rightarrow \rightarrow$ f in measure. 122 5 Moments and Laws of Large Numbers (ii) We say that (Xn)n \in N fulfills the strong law of large numbers if , +1 P lim sup Sn = 0 = 1.2 x 0 2 5/2 2 x 1 0 Fig. Show that X is lattice distributed if and only if there exists a u = 0 such that $|\phi(u)| = 1$. Let Z0 = 1 and inductively define Zn+1 = Zn Xn,i for $n \in N0$. $2p - 1 \blacklozenge 476$ 19 Markov Chains and Electrical Networks 5 4 3 2 1 0 Fig. (8.18) has the properties of the conditional expectation. $n \rightarrow \infty$ (iii) If $A \subset 2\Omega$ is a σ -algebra and if $An \in A$ for every $n \in N$, then $A * \in A$ and $A * \in A$. In particular, we let $\{X \ge 0\}$:= roof We check that the right-hand side in (8.18) has the properties of the conditional expectation. X-1 ([0, ∞)) and define {X ≤ b} similarly and so on. A quick test detects a defective device with probability 95%; however, with probability 95%; however, with probability 10% it gives a false alarm for an intact device. "(iv) = (v)" Let A, A1, A2, . 17.5 Application: Recurrence and Transience of Random Walks 415 As E is finite, there is a y \in E with G(x, y) = ∞ . A Chapter 21 Brownian Motion In Example 14.48, we constructed a (canonical) process (Xt) t $\in [0,\infty)$ with independent stationary normally distributed increments. (iii) Show that if Y is adapted, integrable events form the so-callecter of exchangeable σ -algebra E. The set function μ on A defined by $\mu(A) = (iv)$ (v) (vi) (vii) 0, if A is finite, ∞ , if Ac is fini Proof Let X[°] and Y[°] be two independent Markov chains on E, each with transition matrix p. (iv) Let $k \in N$ and let $\phi : E k \rightarrow R$ be a map. 20.1 Definitions ... 14.2 Finite Products and Transition Kernels 315 Evidently, B((Rd)n) = $\sigma \phi n$ (A1 ×···× An) : A1 , . = (X t, t \in [0, ∞)) of X whose paths are locally (i) There is a modification X Hölder-continuous of every order $\gamma \in 0$, $\beta \alpha$. Define Bi0 = Ai and Bi1 = Aci for $i \in I$. Thereafter, for irreducible aperiodic chains, we state the convergence theorem. A $\subset E$ is called relatively compact if A is compact. Thereafter, for irreducible aperiodic chains, we state the convergence of measures (premeasures, contents) and let (αn) $n \in N$ be a sequence of nonnegative numbers. are i.i.d. (with nontrivial distribution), then trivD $n \rightarrow \infty$ ially $Xn \rightarrow X$ but not $Xn \rightarrow X$ in probability. However, in general, $B \in /F\tau$ since up to time τ , we cannot decide whether X will ever exceed K + 5. n=1 Reflection Check that suprema of lower semicontinuous functions are lower semicontinuous. Example 2.2 (Rolling a die three times) We roll a die three times. In order to illustrate this, assume that X and Y are nontrivial independent real random variables. Proof For the case d = 1 see [19, §20, Theorem 23] or [54, Chapter XIX.2, page 622]., 2m }, and let D = Dm. Hence, by (20.7), ($n \rightarrow \infty 1 + P[A]2$. $n \rightarrow \infty (vi)$ lim μn (A) = $\mu(A)$ for all measurable A with $\mu(\partial A) = \mu(A)$ for all measurable A with $\mu(A) = \mu(A)$ 0. < sn < t and all i1, . However, for the formal proofs of the latter inequalities, we will follow a different route. E (T⁻tK + s - Ts Hence, by Kolmogorov's moment criterion (Theorem 21.42 with $\alpha = 4$ and $\beta = 1/2$), (L[T⁻Kn, n], n \in N) is tight in M1 (C([0, $\infty))$). Proof The claims follow inductively by Theorem 14.25. 703 Subject Index. To this end, let Y be an independent copy of X; that is, a random variable with PY = PX that is independent of X. Lemma 14.7 Let $\emptyset = J \subset I$. We denote by $-br, p \propto -r(-1)k pr(1-p)k \delta k := k (1.17) k=0$ the negative binomial distribution or Pascal distribution or Pasc of the intervals [ak-1, ak]. Reflection Find an example for strict inequality in (17.8). So far, with our machinery we can only deal with conditional probabilities of the type $P[\cdot | X \in [a, b]]$, a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$), a < b (since $X \in [a, b]$). of Lebesgue measure λ (Remark 1.67) to show the assertion first for indicator functions. Define $x * := \inf\{x \in R : P[X * \leq x] = 1\} \in R$. Similarly, for three sets A, B, C $\in A$ with finite content, we have $\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(C) - \mu(A \cap B) - \mu(A \cap B)$ neighboring points x0, x1 \in Zd. Definition 15.1 Let K = R or K = C. Then B : $\Omega \rightarrow C([0, \infty))$ is measurable with respect to A Ω (A, A). Consider now two sets A0, A1 = \emptyset . Au , A1 = \emptyset . Furthermore, show (using Exercise 21.2.2) that E[τ_a , b] = -ab. Sometimes a σ -algebra is also named a σ -field. Define gk = |fnk - f| \land g for k \in N. 10.3 Uniform Integrability and Optional Sampling .. n $\rightarrow \infty$ Concluding, we have $\mu = v$ -lim μ knn . , Xtn)-1 = $\delta x \otimes n-1 \kappa ti+1$ -ti i=0 for any choice of finitely many points 0 = t0 < t1 < . lim inf P A $\cap \tau r-n$ (A) = 0 = $n \rightarrow \infty$ 16
\blacklozenge Reflection Why is τr not mixing if r is rational? Recall that F (x, y) is the probability of hitting y at least once when starting at x. As an example for even n, © The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. If k is large enough that $2-k < \epsilon/2$ and if $n \ge nk$, then the first summand is smaller $n \rightarrow \infty$ meas than 2-k. X is called reversible if there is a π with respect to which X is reversible. The map $F: V \rightarrow R$ is continuous and linear if and only if there is an $f \in V$ with F(x) = x, f * for all $x \in V$. Under a mild consistency condition, the resulting set function can be extended to the whole σ -algebra. The Borel-Cantelli lemma shows that infinitely many of countably many independent events occur jointly with probability either 0 or 1 depending on the summability of the probabilities of the single events. 24.1 Random Measures In the following, let E be a locally compact Polish space (for example, E = Rd or E = Zd) with Borel σ -algebra B(E). That is, (i) holds. (ii) We have $F(\infty) \ge \lim_{n \to \infty} \operatorname{Ent}(\infty)$. Summarising, we have $F(\infty) \ge \lim_{n \to \infty} \operatorname{Ent}(\infty)$. at x0., n M + si-1 = N + i - 1 i ≤ n: xi = 1 k i = 1 xi, i ≤ n: xi = 0 we get N + i - 1 - M - si - 1 N + i - 1 (N - 1)! (M + sn - 1)! N - M - 1 + (n - sn)! = \cdot . + R6-1) - 1.8.2 Conditional Expectations 195 Exercise 8.1.1 (Lack of memory of the exponential distribution) Let X > 0 be a strictly positive random variable and let θ > 0. By the individual ergodic theorem, $0 \to \infty \varepsilon \to p$ almost surely. Then the sets $M \le 1$ (E) and M1 (E) are weakly sequentially compact. In addition, clearly, $H(\nu | \mu) + H(\nu) = -\log \mu(\{x\}) \nu(dx)$. Exercise 8.2.3 (Bayes' formula) Let $A \in A$ and $B \in F \subset A$. Then we can define a probability measure on B(Rn) by $A \mu(B) := \lambda n$ (B) λn (A) for $B \in B(Rn)$ with $B \subset A$. 444 18 Convergence of Markov Chains Proof It is enough to consider the case $\mu = \delta x$, $\nu = \delta y$ for some x, $y \in E$. A Exercise 14.4.3 Show that a nonnegative convolution semigroup is continuous. A 134 5 Moments and Laws of Large Numbers Exercise 5.3.2 Let (Xn) $n \in N$ be a sequence of independent identically distributed $n \rightarrow \infty$ random variables with n1 (X1) . With respect to the image measure $P = P \circ B - 1$ on $\Omega = C([0, \infty))$, the canonical process $X = (Xt, t \in [0, \infty))$ on $C([0, \infty))$ is a Brownian motion. Choose an enumeration of the rational theorem, the sequence numbers $Q = \{q1, q2, q3, ... Hence, we have d fn(t) = n P1 [Xt = n] = n (fn-1 (t) - fn(t))$. 2.2 can be considered as a closed path in the dual graph.) We cite a theorem of Kesten [94]. We give the details. Proof (i) This is a direct consequence of Remark 5.2(ii).*) (ii) By Theorem 5.3(iii), we have E(X - E[X]) = 0 a.s. (iii) Clearly, $f(x) = E[X_2] - 2x E[X] + x 2 = Var[X] + (x - E[X]) = 0$ as direct consequence of integrals is played by the concept of uniform integrability. Now let $I = \emptyset$ and $\pi \in I$. For E := In particular, this implies $\phi(Em \ m \ m \ n \ Em \ (n \rightarrow \infty)$ and $m \ m \) \le \phi(Em \) + \phi(Am \) = \phi(Em \) + \phi(Em \) + \phi(Am \) = \phi(Em \) + \phi(Em \) = \phi(Em \) + \phi(Em \) = \phi(Em \) + \phi(Em \)$ 1.1.3 that Ug is Borel measurable. If $P[A] \in \{0, 1\}$ for any $A \in F$, then E[X | F] = E[X]. (ii) For a real random variable X, the map FX : $x \to P[X \le x]$ is called the distribution function of X (or, more accurately, of PX). (ii) The exponential distribution exp θ for $\theta > 0$. We have got acquainted with some fundamental probability distributions: •••••••• • Bernoulli-distribution Berp on $\{0, 1\}$ binomial distribution bn,p on $\{0, . Theorem 23.8$ The rate function in an LDP is unique. The full strength of the result is displayed in the following examples. As U K is a martingale, Doob's inequality (Theorem 11.2) yields + P sup l=1,...,n |UIK | $\sqrt{>\epsilon}$ n ,) * $\leq \epsilon - 2$ Var Z1K for every $\epsilon > 0$. \in A with An $\downarrow \emptyset$ and $\mu(A1) < \infty$. Definition 19.13 A map I : E × E \rightarrow R is called a flow on E \ A if it is antisymmetric (that is, I (x, y) = -I (y, x)) and if it obeys Kirchhoff's rule: I (x) = 0, for x \in E \setminus A, (19.8) I (A) = 0. Thus we henceforth assume Xn ≥ 0 almost surely for all $n \in N$. Evidently, $\lambda \epsilon, 0 = 1$, and if $\epsilon > 0$ is very small, then $\lambda \epsilon, N/2 = 2\epsilon - 1$ is the eigenvalue with the This shows (5.11). Then $f \tau$, $(X\tau + m)m \in N0 = 1{\tau \leq n} 1{Xn > a} + 12 1{Xn = a}$. Hence there is a constant $C < \infty$ such that $\mu pn - UE T V \leq C \gamma n$ for all $n \in N$, $\mu \in M1$ (E). In addition, f 2 = 0, f * 1/2. We can construct $\tau :=$ the coupling using two independent chains X^{\sim} and Y^{\sim} by defining X := X, $\tau = 1$ and $0 Y_{n} := Y^{\sim}n$, $Xn \to 1$ $n < \tau^{\sim}$, if $n \ge \tau^{\sim}$. For $e \in E$, let $pe \ge 0$ be the probability that e occurs. The second statement follows from Theorem 4.9(i) since $|f + g| \le 4.1$ Construction and Simple Properties 103 |f| + |g|; hence $f + g1 = |f + g| d\mu \le |f| d\mu + |g| d\mu = f1 + g1 + g1 + g1$. Show that D max{X1, . 3b Show that, for all $0 \le a < b$, the map $\omega \to a Xt(\omega)$ dt is measurable. We will show that Cramér's theorem implies that Pn := PSn /n satisfies an LDP with rate n and with good rate function I (x) = $A * (x) := \sup \in \mathbb{R}$ (tx - $\Lambda(t)$). For any $x \in E$, let gx be as in Step 3. Note that Hn depends on D1, . Denote by $N \in \{0, 1, ...\}^*$ * Since $P\pi A \epsilon \cap \tau - n$ (B) - $PA \cap \tau - n$ (B) - Applying Lemma 10.10 to M yields $X\sigma = A\sigma + M\sigma = E[A\sigma + MT F\sigma] = E[A\tau + MT F\sigma$ Polish spaces (like Rd). Further, let Xt be the projection on the tth coordinate. (If G = (V, E) is a planar graph; that is, a graph that can be embedded into R2 without self-intersections, then the vertex set of the dual graph is the set of faces of G., YN be independent random variables with E[Yt] = 0 for t all t = 1, $e \in E$ Takeaways For random variables with second moments, a strong law of large numbers can be shown using the Borel-Cantelli lemma and Chebyshev's inequality first on an subsequence and then on the full sequence. Clearly, the product measure $\mu = \mu 1 \otimes \mu 2$ is a coupling, but in many situations there are more interesting ones. be independent random variables with unknown continuous distribution functions F and F^{*} and with empirical distribution functions Fn and F^{*} n. Thus the Borel-Cantelli lemma belongs to the class of so-called 0-1 laws. $\in E + f$ such that q-1 gn $\uparrow |f| \mu$ -a.e. Define hn = sign(f)(gn) $\in Ef$; hence q gn $q \leq hn f d\mu = F(hn) \leq F p \cdot hn p = F p \cdot (gn q)q-1$. 14 1 Basic Measure Theory (iii) μ is subadditive. 21 and leave this as a warning for the time being. Then g - h is almost everywhere defined and measurable. For $n \in N$, $1 * let Sn := \sqrt{2} i=1$ (Xi $- \mu$). Remark 15.36 For odd moments, the statement of the theorem may fail (see, e.g., Exercise 15.4.4 for the first moment). For example, roulette is such a game., N}, if $j = i - 1 \in \{0, ..., 1 + let Sn := \sqrt{2} i=1 \}$ Then μ is a σ -finite content on A (even a premeasure) since ∞ n=1 (-n, n] = R and $\mu((-n, n]) = 2n < \infty$ for all $n \in \mathbb{N}$. As a corollary to Theorem 7.18, we get the following. Since ψ is strictly convex, in this case, we have $\psi(z) > z$ for all $z \in [0, 1)$; hence $F = \{1\}$. Hence $Z \sim \beta M, N-M$. We want to show that the random variables f (X) and g(X) are nonnegatively correlated. (ii) I (f + g) = I (f) + I (g). For \in S(n) and x = (x1, . Proof For N \in N, define d^{*}N (f, g) := 1 \land d(f (ω), g(ω)) μ (d ω). We conclude that) * f = inf c \in R : P f -1 ((c, ∞)) = 0 P-a.s. " \leftarrow " Assume any I-measurable map is P-a.s. constant. i=1 Indeed, if we let Yn := 1 n n Xi , then (by Example 12.13) (Yn)n \in N is a backwards i=1 martingale with respect to (Fn)n $\in -N$ = (E-n)n $\in -N$ and thus $n \rightarrow \infty$ Yn $\rightarrow Y \infty = E[X1 \ E]$ a.s. and in L1., UN-1) such that P[(X1, See Figs. Then; ($\mu - \nu$)pn; TV $\rightarrow 0$ for all $\mu, \nu \in M1$ (E). Therefore, X and Y coalesce almost surely. Further, let ($p\omega$) $\omega \in \Omega$ be nonnegative numbers., xN) to be a left eigenvector for the eigenvalue λ , the following equations have to hold: $\lambda x = rxk - 1 + (1 - r)xk + 1$ for k = 2, Recursively, define the nth iterate of ψ by $\psi 1 := \psi \psi n := 2, 3$, By the factorization lemma (Corollary 1.97 with f = X and q = Z), there is a map $\phi : E \to R$ such that ϕ is E - B(R)-measurable and $\phi(X) = Z$. $d\nu d\mu \nu 0 \mu$. (10.4) Remark 10.5 If Y and A are as in Example 10.2 then A is monotone increasing since (Xn2) \in L as submartingale (see Theorem 10.1). Then $\mu(A) = A f d\lambda = 0$ for every $A \in A$ with $3\lambda(A) = 0$; hence $\mu 0 \lambda$. (Note that the three neighboring edges of a trifurcation point are in different equivalence classes.) We turn the set $HL := UL \cup TL$ into a graph by considering two points $x \in TL$ and $u \in UL$ as neighbors if there exists an edge $k \in u$ which is incident to x. For a detailed description, see [12]
or [83]. Then there exists a disjoint decomposition of the state space E = d-1 (18.5) Ei i=0 with the property p(x, y) > 0 and $x \in Ei \Rightarrow y \in Ei+1$ (mod d). (20.2) Furthermore, we have $\{Mn > 0\} c \subset \{Mn = 0\} \cap \{Mn \circ \tau \ge 0\} \subset \{Mn - Mn \circ \tau \le 0\}$. For distinct leaves these are distinct points since the leaves belong to disjoint open clusters. Now, de Finetti's theorem states that any infinite and exchangeable σ -algebra. Thus let Y1 , Y2 , . For example, assume that E is partially ordered with a smalles element 0 and a largest element 1 (like the Ising model). (14.17) k=0 For any probability measure μ on E, there exists a unique probability measure μ on E I, B(E) I with the property: For any choice of finitely many numbers 0 = t0 < t1 < / t2 < . i=1 Proof The proof is the same as for Theorem 1.55. 12.1 Exchangeable Families of Random Variables Definition 12.1 Let I be an arbitrary index set and let E be a Polish space. Further, let $M = \sup \epsilon [0,1] |Bt|$, where B is a Brownian bridge., $Un \in U$ such that $n \lambda d$ (Ui) > i=1 1- $\epsilon \lambda (W)$. Then (aX + b) ~ Nau+b, a 2 $\sigma 2$. The maximum and limit (superior) can be interchanged and hence max lim sup $\epsilon \log(a\epsilon i) = \lim \sup \epsilon \log(a\epsilon i) = \lim$ $\leq \lim \sup \epsilon \log \max a\epsilon i N \epsilon \rightarrow 0 a\epsilon i = 1 \leq \lim \sup \epsilon \log(n) + \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \sup \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) = 1, ..., N \epsilon \rightarrow 0 = \max \lim \epsilon \log(a\epsilon i) =$ Brownian bridge. n=2N We conclude $pc \le 23$. Note that F(v) = F(0) = 0 for all $v \in N$ since F is continuous. * Exercise 21.5.5 Consider the coefficients (An $n \in N0$ of the Fourier basis of the construction of Brownian motion. We first define the notion of exchangeability. If there exists a strategy H and a v0 such that $VT = v0 + (H \cdot X)T$, then the trader can sell the claim for v0 (at time 0) and replicate the claim by building a portfolio that follows the trading strategy H. Let $\delta := \sup \inf \sup \mu(Acn, N)$. Hence we have $n \rightarrow \infty P[F1n(x) = y] \rightarrow \pi(y)$ for every y. Lemma 13.5 If E is Polish and $\mu \in Mf(E)$, then for any $\varepsilon > 0$, there is a compact set $K \subset E$ with $\mu(E \setminus K) < \varepsilon$. On the other hand, if she loses, then ir the second game she doubles the stake; that is, $H^2 = 2$ if D1 = -1. Exercise 15.2.1 Let ϕ be the characteristic function of the d-dimensional random variable X. 1.3 The Measure Extension Theorem 35 Later we will see (Corollary 1.84) that B(R) = B(A), where B(A) is the Borel A σ -algebra on A that is generated by the (relatively) open subsets of A. Since B1 \supset B2 \supset . Now we determine the asymptotic growth rate of X. The statement for lim sup Sn is similar. $\Lambda \uparrow Zd$ 448 18 Convergence of Markov Chains 0.7 0.6 Magnetization 0.5 0.4 0.3 0.2 0.1 0 0.84 0.85 0.86 0.87 0.88 0.89 0.9 0.91 0.92 Inverse temperature Fig. Then E[UN] \leq (E[|XN|] + |a|)/(b - a). (ix) Let $\mu \in \mathbb{R}$ and let Σ be a positive definite symmetric d × d matrix. Define the shift operator $\tau: \Omega \rightarrow \Omega$, (ωn) $n \in N0 \rightarrow (\omega n+1) n \in N0 \rightarrow ($ -1 = P. 7.2, we first derive some of the important inequalities (Hölder, Minkowski, Jensen) and then in Sect. Hence $\mu *$ is an outer Step 4 (Closed sets are $\mu *$ -measurable). Proof By the superposition principle, f := f1 - f2 is harmonic on E \ A with $f \equiv 0$. Exercise 1.3.1 Show the following generalization of Example 1.58(iv): A measure $\infty \alpha \delta n xn$ is a Lebesque-Stieltjes measure for a suitable function F if and only if n=1 n: $|xn| \le K \alpha n < \infty$ for all K > 0. Clearly, we have $|m((-\infty, 0))n \le \mu((-\infty, 0))n \ge \mu((-\infty, 0)$ random variable with characteristic function ϕ . 264 12 Backwards Martingales and Exchangeability Theorem 12.14 (Convergence theorem for backwards martingales) Let (Xn)n \in -N0 be a martingale with respect to F = (Fn)n \in -N0. That is, we understand Xs as the initial value of a second Markov process with the same distributions (Px) x \in E. Mainz, Germany June 2020 Achim Klenke v Preface to the Second Edition In the second edition of this book, many errors have been corrected. Then $F = (Fn, n \in N0) = \sigma(Y)$ is the filtration generated by $Y = (Yn)n\in N$ and X is adapted to F; hence $\sigma(X) \subset F$., tN $\in K$ with $K \subset N$ i=1 B δ (ti). The speed of convergence is exponential and the exponential rate function is the relative entropy, be uncorrelated, square integrable, centered random variables and let (an) $n \in N$ be an increasing sequence of nonnegative numbers such that ∞ (log n)2 an-2 Var[Xn] < ∞ . Remark 9.11 Clearly, a stochastic process is always adapted to the filtration it generates. E0 E2 E1 Fig. Which p maximises the entropy under the constraint? For every set $A \subset E$, define $\tau A := \inf\{t > 0 : Xt \in A\}$. In the right figure, the nodes at the ends of xz/zx, xy/yx and yz/zy are split into two nodes and then connected by a superconductor (bold line). Corollary 13.7 The Lebesgue measure λ on Rd is a regular Radon measure. \blacklozenge Lemma 3.5 (Generalized binomial theorem) For $\alpha \in R$ and $k \in N0$, we define the binomial coefficient $\alpha \alpha \cdot (\alpha - 1) \cdots (\alpha - k + 1)$. In order to be able to recycle the terms later in a more generality than is necessary for the treatment of
martingales only. \blacklozenge Example 17.20 (Branching process as a Markov chain) We want to understand the Galton-Watson branching process (see Definition 3.9) as a Markov chain on E = N0. For p > 1, Lp bounded martingales also converge in Lp. (ii) Let $A \subset B$. (P2) The distribution of NI depends only on the length of I: PNI = PNI for all I, I \in I with (I) = (I). Proof Let $\phi = \phi Xk - \mu$. Let Fn denote the distribution function of Sn* and F Φ the distribution function of the standard normal distribution. \$ 18.3 Markov Chain Monte Carlo Method Let E be a finite set and let $\pi \in M1$ (E) with $\pi(x) := \pi(\{x\}) > 0$ for every $x \in E$. (17.5) In this case, the n-step transition kernels $\kappa n = \kappa n - 1 \cdot \kappa 1 = \kappa n - 1 \cdot \kappa n + 1 \cdot \kappa$ $a, b \in [0, \infty)$, $a \leq b$, ((a, b]) := b - a (the length of the interval I = (a, b]). We now assume that the individual opinions may change at discrete time steps. (ii) (fn)n \in N is a Cauchy sequence in Lp (μ). However, there is no number H1 such that H1 X1 = Y1. This suggests that we can use Cb (E) and Cc (E) as classes of test functions in order to define the convergence of measures. For the classical model, we saw (Example 12.29) that the fraction of black balls in the urn converges a.s. to a Beta-distributed random variable Z. Here we are concerned with notions of mixing that lie between these two. In order to underline this, we present the following theorem that will also be useful later. s Hence Xg := $(e-t h(Xt))t \ge 0$ is an F-supermartingale. Hence it remains to show that almost surely N does not assume the value ∞ . If A is a ring, then $\mu(B) = \mu(A) + \mu(B \setminus A)$ for any two sets A, $B \in A$ with $A \subset B$. In particular, $E[W\infty] = E[W0] = 1$. multiple of ν . Then we describe those properties necessary for such a function to qualify as a probability assignment. Below the Curie temperature, these materials are magnetic, and above it they are not. Let Bn := {-n, -n + 1, . To this end, let (a, b], (a(1), b(1)], (a(2), b(2)], . 4 t ~ 2\pi/D 2 2 nt (17.21) 3 ∞ Since 1 t - α dt < ∞ if and only if α > 1, we also have $GY < \infty$ if and only if D > 2. Since $[\omega 1] \supset [\omega 1, \omega 2] \supset ...$, $Cn, mn \in A$. (ii) Let $A = \{E[X | F] < E[Y | F]\} \in F$. As the path $\delta b, t$ has length b | t |, we get the estimate r-1 - z dz $\leq br e - b$ (1 + t 2) $r/2 \rightarrow 0$ for $b \rightarrow 0$. x0, x1 $\in E$ We infer that the entropy of the dynamical system is $h(P, \tau) = -\pi(\{x\})P(x, y)\log(P(x, y))$. 18.5 Equilibrium states of the Ising model on an 800 × 800 grid (black dot = spin +1). By $\mu(A) := \#A \#\Omega$ for $A \subset \Omega$, we define a probability measure on $A = 2\Omega$. n=1 If $g(t) := \mu(\{f \ge t\}) = \infty$ for some t > 0, then both sides in (4.8) equal ∞ . Hence, for $n \in N$ sufficiently large and $k \in N$ with $l(kn+1)/l(kn) \to kn-1 \le k \le kn$, we have $|Sk|/l(kn) \ge 2|Sk|/l(kn)$. By Lemma 1.49, a set $B \subset E$ is $\mu *$ measurable if and only if $\mu * (B \cap G) + \mu * (B \cap$ or $\mu = w$ -lim μn , if $n \to \infty$ f d $\mu n \to f$ d μ for all $f \in Cb$ (E). be measurable with fn $\geq f$ a.e. for all $n \in N$. + E[Xn] = Sm. The next step is to show convergence of finite-dimensional distributions., ωn] and P = ((1 - p)\delta - 1 + p\delta 1) \otimes N is the product measure., lk) is of particular importance. (ii) The following equivalences hold: q 1 z 1 (=) ∞ kpk > 1. Proof As $B(\Omega) = \sigma(\tau)$ and by Theorem 1.81, it is sufficient to show that f - 1 (A) $\in \sigma(\tau)$ for all $A \in \tau$. For $n \in N$, let $\mu n = \lambda$. D Assume PYn = N0,1/n for all $n \in N$. First assume that $C \subset R$ is closed. Let μ be a probability measure on Σ with $\mu(\{x\}) > 0$ for any $x \in \Sigma$. Substituting x = t, we obtain ∞ n' (E Yn2 = 2x P[|Yn| > x] dx $\leq 2x P[|X1| > x] dx$. Proof We carry out the proof by induction on $n \in N0$. Using the Cramér-Wold device (Theorem 15.57), this implies) * $n \rightarrow \infty$) * Lx Z^{*} tn1 , Z^{*} tn2 \rightarrow Lx Yt1 , Yt2 . Let A := n=1 An . Reflection The proof of the previous theorem made use of the fact that (Xn+)n \in N and (Xn-)n $\in N$ are uncorrelated families. 1 – (1 – p)z (3.6) 88 3 Generating Functions By the generalized binomial theorem (see Lemma 3.5 with $\alpha = -n$), Theorem 3.3 and (3.6), we have pn (1 - (1 - p)z)n ∞ -n = pn (-1)k (1 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ X1 (z)n = k=0 = ∞ - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ Y1 (z)n = k=0 = \infty - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ Y1 (z)n = k=0 = \infty - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ Y1 (z)n = k=0 = \infty - bn, p ({k} zk (2 - p)k zk k ψ Y (z) = ψ Y1 (z)n = k=0 = \infty - bn, p ({k} zk (2 - p)k zk (2 - p and let v be the Lebesque measure on R. (iii) For the Moran model (Example 17.22), use the explicit form (17.12) of the square variation process to compute the transition matrix. Indeed, we show that Pp [N \geq 3] = 0. "Separable" means that there exists a countable dense subset. Consequently, Yn := Y^Sn + X n - Sn , n \in N, is also a random walk with transition matrix p. 158 6 Convergence Theorems "(iii) \Rightarrow (i)" Let f be the limit in measure of the sequence (fn)n \in N. + Xn) \rightarrow Y almost surely for some random variable Y. Proof We only have to show the triangle inequality. For one coordinate, however, which moves only with probability 1/D and thus has variance 1/D, the probability of being back at the origin at time 2n is approximately $(n \pi/D) - 1/2$. Then PX is uniquely determined by the distributions of either of the families (If1, densities); hence we have P[T $\infty \infty$ Now let r Rt := sup n \in N : T1r + . We give a probabilistic proof for this formula. In this case, in general, the total mass $\mu(\Omega)$ is not uniquely determined by the values $\mu(E)$, E \in E; see Example 1.45(ii). Conclude D that (X1, X2) = (Y1, Y2) and thus PX = PY. Having defined Wn, we choose a relatively open set Ln \supset Wn and define Wn+1 := Ln \cup Un+1 . 1.1. σ -algebra σ - \cup -stable ring \cup -sta but close in which sense? A set $A \subset E$ is called totally bounded if, for any n $\epsilon > 0$, there exist finitely many points x1,. In the next theorem, we collect the characteristic functions for some of the most important distributions. Let $L = maxe \in E$ (e). 23.3 Sanov's Theorem 599 Recall that the entropy of μ is defined by $H(\mu) := -\log \mu(\{x\}) \mu(dx)$. Then |f| $|p \leq f \propto almost everywhere;$ hence $p p |f| p du \leq f \propto du = f \propto u(\Omega) < \infty$. The stochastic order belongs to the class of so-called integrals with respect to a certain class of functions (here: monotone increasing and bounded) are ordered. (17.10) $z \in E$ Definition 17.16 A matrix (p(x, y))x, y \in E with nonnegative entries and with p(x, y) = 1 for all $x \in E y \in E$ is called a stochastic matrix on E. For a survey on different orders of probability measures, see, e.g., [120]. Clearly, Zk is a nsystem and σ (Zk) = σ (Xj, j \in Ik). The preceding theorem says, in particular, that we cannot find any locally bounded gambling strategy that transforms a martingale (or, if we are bound to nonnegative gambling strategies, as we are in real life, a supermartingale) into a submartingale. Exercise 21.2.1 Let B be a Brownian motion and let λ be the Lebesgue measure on $[0, \infty)$. (21.31) We now define S⁻ n as S n but linearly interpolated: 1 S⁻tn = $\sqrt{\sigma}$ 2n nt ! Yi + i=1 tn - tn! Y $\sqrt{\sigma}$ 2n nt ! +1 . 1 - 2- γ Analogously, we obtain $|Xs(\omega) - Xu(\omega)| \le 2-\gamma n$ ($1 - 2-\gamma)-1$, and thus $|Xt(\omega) - Xs(\omega)| \le 2-\gamma n$. That is, (18.12) holds. Applying the continuous. Exercise 17.6.1 Consider the Markov chain from Fig. \blacklozenge Reflection If X is a martingale with respect to some filtration F, then X is adapted to any larger filtration $F \supset F$ but it is not necessarily an F-martingale., Xn-1 = N + n-1 Inductively, for x1, . 5.3 Strong Law of Large Numbers . be independent and identically distributed with $E[X1] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] =
\sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = \sigma 2 \in (0, \infty)$ and $\gamma := E[|X1|] = 0$ and $\gamma :=$ exactly one edge; that is, by the edge in E that separates the two faces. It follows D that (XTt) t $\geq 0 = (Yt)t \geq 0$. Examples include the Gamma distribution vt = Poit, the negative binomial distribution vt = Poit, the negative binomial distribution vt = Poit, the negative binomial distribution vt = Caut and others (compare Theorem 15.13 and Corollary 15.14). In this case, $f 2 = \frac{\varphi}{(2\pi)d/2}$. Hence, if we define qe := 2 - l(e), then (note that #CL(e) = 2L - l(e), then (note that #CL(e) = 2L - l(e), then (note that #CL(e) = 2L - l(e)). between the moments of a real random variable X and the derivatives of its characteristic function ϕX ., $\omega k] \cap Cn$, in = \emptyset for infinitely many $k \in N$. Furthermore, $\alpha k = E[Xk] = x \nu k$ (dx) for $k \ge 1$. We collect some simple properties of characteristic functions. Theorem 9.16 Let I be countable. Then $\delta \omega$ is a probability measure on any σ -algebra $A \subset 2\Omega$. Hence $F = \{r, 1\}$ and $q = \min F = r$. 31 Cauchy's principal value of the integral -1 f (x)dx is defined as $\lim n \to \infty -1/n -1$ 1 dx + x 1 1/n 1 dx x = 0. "Uniqueness" Let)x, f* =)x, g* for all $x \in V$. \blacklozenge Example 11.16 (Voter model, due to [28, 75]) Consider a simple model that describes the behavior of opportunistic voters who are capable of only one out of two opinions, say 0 and 1. Then $|f(t) - f(s)| \le 2f \propto |t - s| \gamma \le C |t - s| \gamma$ Coupling In many situations, for the comparison of two distributions, it is helpful to construct a product space such that the two distributions are the marginal distributions but are not necessarily independent. 84 2 Independence 0 Fig. • Theorem 9.32 (i) X is a supermartingale if and only if (-X) is a submartingale. If Ω is uncountably infinite, this is wrong in general. $j \in J$ This is obvious since $\{x\} : x \in E \cup \{\emptyset\}$ is a π -system that generates 2E, thus -1 Xi ($\{xi\}$) : $xi \in E \cup \{\emptyset\}$ is a π -system that generates σ (Xi) (Theorem 1.81). 498 20 Ergodic Theory Lemma 20.13 (Hopf's maximal-ergodic lemma) Let X0 \in L1 (P). $x \in V$ Nx is a P-null set and (8.8) holds for every 8.2 Conditional Expectations 201 The map $x \rightarrow D + \phi(x)$ is right continuous (by Theorem 7.7(iv)). Now let $p, p \in [1, \infty)$ with p < p and let $f \in Lp(\mu)$. Then $\phi(t) = P[X1 = 0]$. 2 Remark 15.39 If we prefer to avoid the continuity theorem, we could argue as * * 2 follows: For every K > 0 and $n \in N$, we have $P[|Sn| > K] \leq Var[Sn]/K = 2 1/K$; hence the sequence PSn* is tight. In the definition of p, the distribution π appears only in terms of the quotients $\pi(y)/\pi(x)$. If we denote $A = \{X \in B\}$ for $\in S(n)$, then $En = \{A : A = A \text{ for all } \in S(n)\}$. A measurable map $f: E \to R$ is called uniformly integrable with respect to $(\mu n)n\in N$, if inf sup a>0 $n\in N$ {|f|>a} |f| d $\mu n = 0$., state that the $n\to\infty$ sequence of averages converges a.s. to the expected value, n-1 ni=1 Xi $\rightarrow E[X1]$. As Example 1.39 We aim at extending the volume $\mu((a, b]) = ni=1$ (bi -ai) that was defined on the class of rectangles A = {(a, b]: a, b \in Rn, a \le b} to the Borel σ -algebra B(Rn). 310 14 Probability Measures on Product Spaces Finally, for pairwise disjoint sets A1, A2, . Hence, instead of giving a proof, we refer to the textbooks on measure theory (e.g., [37]). We draw n of these balls without replacement but respecting the order. First, consider the case where p(0, x) = 3-d for all $x \in \{-1, 0, 1\}d$. Proof Without loss of generality, we can do the computation with b = 1e; that is, with the natural logarithm. Its distribution function is FX (x) = $\int \{0, 1 - p, | 1, \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x \in [0, 1], \text{ if } x < 0, \text{ if } x \in [0, 1], \text{ if } x \in [0, 1]$ variable taking values in the space of Radon measures on a set E. Proof This follows directly from Theorem 17.36. As α approaches 1/2, the distribution $\mu\alpha$ has less and less moments. 10.1 Doob Decomposition and Square Variation . 26.1 Strong Solutions . Define Zn := a.s. supk $\geq n$ [Xk - X]. < tm = 1. In the following, let (E, τ) be a Polish space with Borel σ -algebra E. be i.i.d. random variables with values in Σ and distribution *) PX1 = μ . Takeaways For ergodic dynamical systems, averages over trajectories coincide (ergodic theorems). Let En := ni=1 Ω i, n \in N, and E0 = \emptyset . Hence we can define $u := z/F(z) \in W \perp$. Further, we defined the spaces of functions where these expressions are finite: Lp (Ω , A, μ) = Lp (μ) = product of the spaces (Ω i, i \in I), or briefly the product space. In Exercise 18.2.4, an example is studied that shows that spacial homogeneity cannot easily be dropped if we want to have a successful coupling. Show that a probability measure π on E is an invariant distribution for X if and only if $x \in I$ ($\{x\}$)q(x, y) = 0 for all $y \in E$. Then μ is a Radon measure if and only if $u(K) < \infty$ for any compact $K \subset E$. Define $Xi = \#\{k \le n : Yk \le pi\}$, i = 1, 2. 321.9 Pathwise Convergence of Branching Processes (branching processes). Then, by Sanov's theorem for open $U \subset Rd$, lim inf $n \rightarrow \infty$ 1 log PSn /n (U) = lim inf log $P\xi n$ (X) m-1 (U) $n \rightarrow \infty n n \ge -$ inf $I\mu m-1$ (U) = - inf $I^{\sim}(U)$. This makes the notion quite a bit more flexible. As each An could still contain "portions with $n \rightarrow \infty F - 1$ (u) at every point of continuity u of F - 1. Evidently, supn $\in N$)X* $\tau K \wedge n \le K$ almost surely. If $A \in G$

and $f = 1\Omega 1 \times \Omega 2 \setminus A$, then clearly $\omega i \rightarrow f(\omega 1, \omega 2) \mu 3 - i(\omega 3 - i) = \mu 3 - i(\omega 3$ null sets. Lemma 1.50 M($\mu *$) is an algebra. \bullet be the Exercise 13.2.3 Let E = R and $\mu n = n1$ nk=0 $\delta k/n$. Show that $\mu \varepsilon := P \varepsilon X \lambda/\varepsilon$ satisfies an LDP with good rate function I (x) = x log(x/\lambda) + $\lambda - x$ for $x \ge 0$ (and $= \infty$ otherwise). By Remark 8.26, it suffices to check this for rectangular cylinders with a finite base $A \in Z R$ since Z R is a π -system that generates B(E) ØI. If $f \ge 0$ or $f \in L1$ ($\mu \otimes \kappa$), then $\Omega 1 \times \Omega 2$ f d($\mu \otimes \kappa$) = f ($\omega 1$, $\omega 2$) $\kappa(\omega 1$, d $\omega 2$) $\Omega 1 \mu(d\omega 1)$. 17.6 Invariant Distributions 427) * Hence, for every $n \in N0$ (since Py $\tau x 1 < \infty = 1$ for all $y \in E$), 1 = n) * * Pn $\sigma xn = k + Pn \sigma xn = -\infty k = 0 = \pi({x}) n ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n - k + 1 + Pn \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty ' ((Px \tau x 1 \ge n + 1 k = 0 n \rightarrow \infty - \rightarrow \pi({x}) \infty) \otimes \pi({x}) \otimes \pi({x}$ $\pi(\{x\})$ Ex $\tau x 1$. \bullet n-1 f $\circ \tau k = 0$ 1 n-1 n k=0 [] f dP P-a.s., n $\to \infty$ f $\circ \tau k \to 3$ f dP, then P (A) = 1 and Q(A) = 0., pn \in [0, 1]. Here equality of events (mod P) means that the events differ at most by a P-null set (see Definition 1.68(iii)). $y \in E$ If we define p(x, y) := $\pi(\{x\}) = C(x)$. Hence it suffices to show that for fixed $\omega 0 \in C(x)$. Ω , the map $\omega \rightarrow d(\omega 0, \omega)$ is A-measurable. If we know that B has occurred, it is plausible to assume the uniform distribution on the remaining possible outcomes; that is, on {1, 2, 3}. It could be objected that this argument works only for probability measures. Example 13.22 If F is the distribution of a probability measure on R and Fn (x) := F (x + n) for $x \in R$, then (Fn)n \in N converges pointwise to 1. Equality holds if and only if there are a, b, $c \in R$ with |a| + |b| + |c| > 0 and such that aX + bY + c = 0 a.s. Proof The Cauchy-Schwarz inequality holds for any positive semidefinite bilinear form and hence in particular for the covariance map. The (time-homogeneous) Markov property of a process means that, for fixed time t, the future (after t) depends on the past (before t) only via the present (that is, via the value Xt). (i) Show that $pc \ge 1/2$. Hence we have to compute the spectrum of p. (ii) If $n \in N$ and $X \in Ln(P)$, then the quantities ' (mk := E Xk, ' (Mk := E |X|k for any k = 1, . (6.6) Here we define $f - fnk 1 = \infty$ if $f \in L1(\mu)$. Hence the random variables $p Y n := (X \tau n (e)) e \in E0$, $n \in Z$, are independent and identically distributed (with values in $\{0, 1\}E0$). Define $Y = (Yn + 1)n \in Z$. If A is an algebra, then in (ii) for any $A \in \sigma$ (A), we even have inf $B \in A \mu(A B) = 0$. We start with some preparatory lemmas. This measure μ is called the uniform distribution on A and will be denoted by UA := μ . i=1 Then ST \in L1 (P) and E[ST] = E[T] E[X1]. Rather, the observations are aspects of the single experiments that are coded as measurable maps from Ω to a space of possible observations. Corollary 14.27 Let $n \in \mathbb{N}$ and let $(\Omega_i, A_i), i = 0, .$ Hence, it, $x^* - \Lambda(t) \leq \inf H(\nu | \mu) \nu \in \mathbb{E}x$ with equality if $\nu t \in \mathbb{E}x$. 611 616 627 25 The Itô Integral . For details, see, e.g., [90]. \blacklozenge Definition 1.73 Let (Ω , A, μ) be a measure space and $\Omega \in A$. \leq th . $\in A$, \bullet \-closed under complements if Ac := $\Omega \setminus A \in A$ for any set A $\in A$. Finally, we have shown that the results for i.i.d. random variables that we had already fit in this framework. (ii) Let Jn be the number of connected subgraphs of T that contain 0. V gn. Example 9.4) Symmetric simple random walk X on Z is a square integrable martingale. Rather, the number of balls that we add varies from time to time. 20.3 Examples 501 Now let $A \in I$ be invariant. If we let $s = e - \lambda$, then we get the Laplace transform of Zn, Ei $[e - \lambda Zn] = \psi$ (n) $(e - \lambda)i$. Proof Without loss of generality, assume I = N. (15.9) 15.5 The Central Limit Theorem 363 n $\rightarrow \infty$ In particular, Tn := $(Y1 + . (vi) Let E be a finite nonempty set and let <math>\Omega$:= E N be the set of all E-valued sequences $\omega = (\omega n)n \in N$. This gives a good candidate for μ . Theorem 2.19 Let E be a finite set and let (pe)e E be a probability vector on E. For $\omega \in N$, define $F^{\sim}(\cdot, \omega) = F0$, where F0 is an arbitrary but fixed distribution function., N -1} and the transition matrix given by p(k, k + 1(mod N)) = 1 for all k. Verbally, each step of the chain with transition matrix p can be described by the following instructions. Proof The statements follow from the elementary theory of power series. The computer time needed for this is at least of the order of the size of the random walk on ZD .) Use Theorem 17.52 to show that X is positive recurrent if and only if G is finite. -1/2 1 Here Cx is a normalising constant and $\mu x = \mu 1 + \sigma 12$ (x $-\mu 1 - \mu 2$) 2 and $\sigma 12 + \sigma 2 \sigma x^2 = \sigma 12 \sigma 22 \sigma 12 + \sigma 22$. Now [at , b] is compact and ∞ (a(k), bt (k)] \supset (a, b] \supset [at , b], k=1 0 whence there exists a K0 such that K k=1 (a(k), bc (k)) \supset (ac, b]. It is called stable if each of the summands has the same distribution as bn + X/an for some sequences (an) and (bn). k i \in I 0 1 N for all $k \in N$. (iv) If X, $Y \in L2$ (P), then we define the covariance of X and Y by) * Cov[X, Y] := E X - E[X] Y - E[Y]. We define :=)z, w*/w2 and get $y+w \in W$; hence $c_2 \leq x - (y + 1)$ w)2 = z2 + 2 w2 + 2)z, w* = c2 - 2 w2 . Let $\Omega = \Omega + \Omega - be$ a Hahn decomposition with respect to $\phi_i = \phi_i + \phi_i = 0$ and let $\Omega = \Omega_i + \Omega_i - be$ a Hahn decomposition with respect to $\phi_i = \phi_i + \phi_i = 0$. Clearly, there is an $n \in N$ such that F1 , $n \rightarrow \infty$ 490 19 Markov Chains and Electrical Networks Proof (i) Let $\tau N := inf n \in N0$: $Xn \in \{-N, N\}$ = E Y - E[X | F] For the case $E[E[X | F]_2] < \infty$, we are done. Similarly, we define locally Hölder- γ -continuous paths and so on. Let F - 1 (u) = inf{x $\in R : F(x) \ge u$ }, u $\in (0, 1)$, be the left continuous inverse of F (see the proof of Theorem 1.104). How quick is the convergence in (18.11)? For finite families, this is not true. (1.1) Hence A := ∞ n=0 An is a semiring but is not a ring (if #E > 1). n=1 By assumption, A $\cap E \in E$ if A, $E \in E$; thus $E \subset DE$ if $E \in E$, k (with the restriction b1 + . 10.1 Doob Decomposition and Square Variation Let X = (Xn)n \in N0. n $\rightarrow \infty$ In other words, limes inferior is the event where eventually all of the An occur. For any $\varepsilon > 0$ and $n \in N$, define the set $Bn \in \{fn \geq (1 - \varepsilon) g\}$. (For example, choose the ε -balls centered at the points of a countable subset and let ε run through the positive rational numbers.) A compact metric space is always separable (simply choose foreach $n \in N$ a finite cover $Un \subset \tau$ comprising balls of radius n1 and then take $U := n \in N$ Un). 18.4 Computer simulation of the magnetization curve of the Ising model on a 1000 × 1000 grid. -; hence indeed b - = CPoi. Example 2.4 Let E be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the
individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in E$ be a finite set (the set of possible outcomes of the individual experiment) and let (pe) $e \in$ integrable random variables. ♦ Corollary 8.17 (Conditional expectation as projection) Let F ⊂ A be a σ algebra and let X be a random variable with E[X2] < ∞. 124 5 Moments and Laws of Large Numbers We present a probabilistic proof of this theorem. Further, show that strict inequality can hold in the upper bound (LDP 2)., xn) ∈ Rn , for some integrable function $f: Rn \rightarrow [0, \infty)$, then f is called the density of the distribution. n=1 Hence $A \in DE$..) $\in \Omega$. We have $D - \phi(x) = D + \phi(x)$ at all points of continuity of $D - \phi$ and $D + \phi$. Definition 15.40 For every $n \in N$, let $kn \in N$ and let Xn, 1, ... Theorem 18.15 Assume that q is irreducible and that for any $x, y \in E$, we have q(x, y) > 0 if and only if $q(y) = D + \phi(x)$ at all points of continuity of $D - \phi$ and $D + \phi$. x > 0. If she wins, then she does not bet any money in the subsequent games; that is, Hn = 0 for all $n \ge 2$ if D1 = 1., $Xn] \rightarrow P\pi [X \in A\sigma (X0, X1, ..., to be sent by e-mail to ., <math>E[Xn]$. For simplicity, as the time interval we take [0, 1] instead of $[0, \infty)$. This is reflected by the above calculation that shows that, for $t \ge \theta$, the exponential moments are infinite. Proof Let $D = J - I \equiv 0$ be the difference of the flows. 6 $\delta e \in \{1, ..., 6\}$ Furthermore, let $An = \{\omega \in \Omega : \omega n = 6\}$ be the event where the nth roll shows a six. Use the star-triangle transformation to remove the lower left node (left $1 = \delta/R1 = 5$, in Fig. Show that X is in if and only if there exists a $C < \infty$ such that $|E[XY]| \leq C Y q$ for any bounded random variable Y. By Remark 2.15, the random variables Sn and $1{T = n}$ are independent for any $n \in N$ and thus uncorrelated. However, for high temperatures (small β), we can approximate $m\beta$, h using the approximation tanh($\beta(m + h)$) $\approx \beta(m + h)$. We have also reformulated the elementary conditional expectation and highlighted those of its properties that allow for a generalisation to conditional expectations given σ -algebras. 6.3 Exchanging Integral and Differentiation .. At the other end of the spectrum is the case where X and F are independent; that is, where knowledge of F does not give any information on X. It follows that E[XY] = $z P[XY = z] z \in \mathbb{R} \setminus \{0\}$ $x \in \mathbb{N}$ $x \in \mathbb{N}$ $x \in \mathbb{N}$ $y = z P[X = x, Y = z/x] x xy P[X = x] P[Y = y] y \in \mathbb{N}$ $x \in \mathbb{N}$ $y P[Y = y] y \in \mathbb{R}$ and construct a contradiction. Show that, for any two starting points, the independent coalescent coupling is successful. Proof By Kolmogorov's 0-1 law (Theorem 2.37), T is trivial., k} is finite. l=0 Define AN,n = ni=1 AN,n,i, $AN = lim infn \rightarrow \infty AN,n$ and $A = \infty N=1 AN$. \blacklozenge Takeaways A Markov processes in continuous time and with discrete state space can be described by its jump rates (q-matrix). Let p be the transition matrix of the random walk on E that stays put with probability $\epsilon > 0$ and that with probability $1 - \epsilon$ makes a jump to a randomly (uniformly) chosen neighboring site. Manifestly, S1 = X0 and Mn $\circ \tau \ge 0$ and hence also (for k = 0) X0 \ge S1 - Mn $\circ \tau$. Theorem 19.20 (Conservation of energy) Let A = A0 \cup A1, and let I be a flow on E \ A (but not necessarily a current flow; that is, Kirchhoff's rule holds but Ohm's rule need not). (ii) Let n \in N and assume i1, . Mathematically, we say that a family of random variables is exchangeable if the joint distribution does not change under finite permutations. Then X is adapted s=1 220 9 Martingales and integrable, and E[Yr Fs] = 0 for r > s. For $n \ge m0$, we thus have $p ~ Y^{-}$) into the diagonal Now define the stopping time τ of the first entrance of (X, D := {(x, x) : x \in E} by $\tau := \inf n \in I$ NO: $X^n = Y^n$. In order to describe such processes formally, we introduce the following notion. (In a more general framework, the Gibbs sampler and the same method.) For states x and y that differ only in the ith coordinate, we have (since x-i = y-i) $\pi(x) = \pi(x) = \pi(x) = \pi(x)$ $\pi(x) = \pi(y)$ $qi = \pi(y)$ p(y, x). = sup $\alpha(C)$: $C \in C$ with $C \subset i=1$ i=1 13.3 Prohorov's Theorem 299 Step 3 (σ -subadditivity of $\mu *$). Then $u \leq s < u + 2 - n$ and $u \leq t < u + 21 - n$ and hence bi (t - u) = 0 for i < n. Hence, it is assumed that each individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently and uniformly at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individual of the new generation chooses independently at random one individua preceding generation as ancestor and becomes a perfect clone of that. (17.16) $x \in E$ Then q is the Q-matrix of a unique Markov process X. By Weierstraß's approximation theorem, there exist polynomials fn of degree at most n such that $n \rightarrow \infty$ fn $-f \propto -\rightarrow 0$, where $f \propto := \sup\{|f(x)| : x \in [0, 1]\}$ denotes the supremum norm of $f \in C([0, 1])$ (the space of continuous functions $[0, 1] \rightarrow R$). (iv) The set of all finite measures on (Ω, A) is denoted by Mf $(\Omega) := Mf(\Omega, A)$. (21.1) ϕ is called locally Hölder-continuous of order γ if, for every $t \in E$, there exist $\varepsilon > 0$ and $C = C(t, \varepsilon) > 0$ such that, for all s, $r \in E$ with d(s, t) < ε and d(r, t) < ε , the inequality (21.1) holds. Assume that f1 and f2 are harmonic on $E \setminus A$. Hence, the map $g(x^{-} + 2KZd) = g(x)g^{-}$: $E \rightarrow C$, is well-defined, continuous and bounded. Hint: Use the regularity of Lebesgue measure (Remark 1.67). As an alternative to the Metropolis chain, we consider a different procedure to establish a Markov chain with a given invariant distribution. Theorem 15.52 (Berry-Esseen) Let X1 , X2 , . To this end we need a table of the Morse code as well as the frequencies of the letters in a typical text. The distribution PY of Y := 2X - 1 is sometimes called the Rademacher distribution with parameter p; formally Radp = $(1 - p)\delta - 1 + p\delta + 1$. $n \rightarrow \infty$ (viii) $\mu = v$ -lim μn and $\mu(E) \ge \lim \sup \mu n$ (E). Define Y = max(X, a). (iv) The class of finite unions of bounded intervals is a ring on $\Omega = \mathbb{R}$ (but is not an algebra). Indeed, the sets $(-\infty, y]$, $y \in \mathbb{R}$ form an \cap -stable generator of B(\mathbb{R}n) and $x \to \kappa(x, (-\infty, y]) = \mu((-\infty, y - x])$ is left continuous and hence measurable. Recall that the support of a real function f is f - 1 ($\mathbb{R} \setminus \{0\}$). Proof See, for example, the book of Grimmett [63, pages 287ff]. Then there exists a unique probability measure μ on σ (A) = B(Ω) such that $\mu([\omega 1, . Let F = (Fn)n \in N 0 = \sigma((Xn)n \in N . a, b \in Q a 0, let <math>\tau K := \inf\{n \in N : X^*n+1 \ge K\}$. Show that 3 P[A|F] dP P[B|A] = 3B. If in addition Ω is finite, then A is finite. \blacklozenge We introduce some further terms. Proof As $\mu *$ is subadditive, the other inequality is trivial. For every interior point $x \in I$, let $D + \phi(x)$ be the maximal slope of a tangent of ϕ at x; i.e., the maximal number t with $\phi(y)
\ge (y - x)t + \phi(x)$ for all $y \in I$ (see Theorem 7.7). We denote by $B(\Omega, \tau) = \sigma(\tau)$ the Borel σ -algebra on (Ω, τ) . 80 2 Independence The cases p = 1 and $\theta(p) = 0$ (hence in particular the case p = 0) are trivial. (ii) There is an $\alpha \in \{0, 1\}$ such that the family (Bi\alpha) is called the Doob decomposition. Then P[X \ge n] = P[[\omega10, . Then \phi T V := sup \phi(A) - \phi(\Omega \A) : A \in $= \phi(\Omega +) - \phi(\Omega -) = \phi + (\Omega) + \phi - (\Omega)$ defines a norm on M± (Ω , A), the so-called total variation norm. This example suggests that we also have to make sure that no mass "vanishes at infinity". k=1 In particular, if A = A1 A2, one similarly gets $\mu(A^2 +) - \phi(\Omega -) = \phi + (\Omega) + \phi - (\Omega)$ defines a norm on M± (Ω , A), the so-called total variation norm. This example suggests that we also have to make sure that no mass "vanishes at infinity". k=1 In particular, if A = A1 A2, one similarly gets $\mu(A^2 +) - \phi(\Omega -) = \phi + (\Omega) + \phi - (\Omega)$ defines a norm on M± (Ω , A), the so-called total variation norm. This example suggests that we also have to make sure that no mass "vanishes at infinity". k=1 In particular, if A = A1 A2, one similarly gets $\mu(A^2 +) - \phi(\Omega^2 +) + \mu(A^2 +$ $h \in N$ be i.i.d. random vectors with E[Xn,i] = 0 and E[Xn,iXn,j] = Cij, i, j = 1, with $In \sim UA$ and $Un \sim U[0,1]$. be events and define $A * = \lim \sup An$. Denote by λ the Lebesgue measure on R. A map $\phi : G \rightarrow R$ is called convex if for any two points x, $y \in G$ and any $\lambda \in [0, 1]$, we have $\phi \lambda x + (1 - \lambda)y \leq \lambda \phi(x) + (1 - \lambda)\phi(y)$. By $n \rightarrow \infty$ assumption, ϕPnk $-\rightarrow$ fk pointwise for some function fk that is continuous at 0. Clearly, (18.5) holds. Since sup S = T $\infty \propto r$ s P[T $\infty = T \propto 1$] = 0. The opposite implications hold only under an additional condition of summability (see Theorem 6.12). In this case, the process X of partial sums Xn = Y1 + . In this chapter, we first lay the foundations for the treatment of general stochastic processes. Let + Rw := ∞ Rw (i, i + 1) = i = 0 ∞ i 1 = k Cw (i, i + 1) for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|fi(t) - fi(s)| < \varepsilon$ for all $i \in I$ and all $s, t \in E$ with $d(s, t) < \delta$. $\sqrt{\varepsilon} 2$ no no 2 (Note that $|un - vn| \le |u - v| \cdot n \cdot max(|u|, |v|)n - 1$ for all $u, v \in C$.) 15.5 The Central Limit Theorem 15.38 (Central Limit Exercise 15.6.2 (Cholesky factorization) Let C be a positive definite symmetric real d × d matrix. 19.14 Graph with enumerated nodes. However, we would like to define for every $x \in E$ a probability measure $P[\cdot | X = x]$ such that for any $A \in A$, we have P[A|X] = P[A|X = x] on $\{X = x\}$. Let (fn)n $\in N$ be a Cauchy sequence in measure in E; that is, for any $A \in A$, we have P[A|X] = P[A|X = x] on $\{X = x\}$. Let (fn)n $\in N$ be a Cauchy sequence in measure in E; that is, for any $A \in A$, we have P[A|X] = P[A|X = x] on $\{X = x\}$. $A \in A$ with $\mu(A) < \infty$ and any $\varepsilon > 0$, we have $\mu A \cap \{d(fm, fn) > \varepsilon\} \rightarrow 0$ for m, $n \rightarrow \infty$. In order to compute the latter quantity, we first determine the cardinality #Pn. Clearly, $q \ge p0$. Therefore, $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$ and $\lim s - 1 P[Xt + s = n + 1|Xt = n] = n2 s \downarrow 0$. hence $\lim s - 1 P[Xt + s = m|Xt = n] - I(m, n) = q(m, n) s \downarrow 0$ for all m, $n \in N$. We consider two edges in KL as equivalent if there exists a path in BL along open edges that does not hit any trifurcation point and which joins at least one endpoint of each of the two edges. This is the starting point for many conclusions. By Theorem 19.6, this implies f (x0)) = f (y) = 0 contradicting the assumption. (iii) A map X : $\Omega \rightarrow \text{Rd}$ is A - B(Rd)-measurable if and only if X-1 (($-\infty, a$]) \in A for any $a \in \text{Rd}$. & Exercise 1.3.2 Let Ω be an arbitrary element. For n = 0, the claim is trivially true. TV We summarize the connection between aperiodicity and convergence of distributions of X in the following theorem. That is, it is enough to show that n kk $(0, \phi_n (A1 \times \cdots \times An)) = P(S1, \dots, Sn)$ $(\phi_n (A1 \times \cdots \times An)) = P(S1, \dots, Sn)$ $(\phi_n (A1 \times \cdots \times An)) = P(S1, \dots, Sn)$ $(\phi_n (A1 \times \cdots \times An)) = P(S1, \dots, Sn)$ every $n \in N$, the family $(Xn,l) = 1, \dots, kn$ is independent, • centered if $Xn,l \in L1$ (P) and E[Xn,l] = 0 for all $n \in N$. By (8.17), $P[Z1 \in \cdot |Z1 + Z2 = x] = N\mu x$, σx^2 for almost all $x \in R$. Then $E[W\infty] > 0 \iff E[W\infty] > 0 \iff E[W\infty] = 1$ $E[X1,1 \log(X1,1)+] < \infty$. Proof If ϕ_n is the CFP of $\mu_n \in M1$ (R), then $e_n(\phi_n - 1)$ is the CFP of CPoirn μ_n . Define B1 = A1 and Bk = Ak \ k-1 (Ak \ (Ak \ Ai = i = 1 k-1 (Ak \ (Ak \ Ai = i = 1 k-1 (Ak \ (Ak \ Ai = i = 1 k-1 (Ak \ Ai = i = 1 k-1 (Ak - Ai)))) for k = 2, ..., \tau n-1 (i) = i +)r* for i = 0, ... Kirchhoff's rule says that the flow is divergence-free and that the flow is din the flow is din the flow is divergence-free and that the f via Markov semigroups) Let ($\kappa t : t \in I$) be a Markov semigroup on the Polish space E. This implies Ln (ϵ) $\rightarrow 0$, where Ln (ϵ) = n *) 2 E Xn, l 1{ $|Xn, l| > \epsilon$ } is the quantity of the Lindeberg condition (see (15.6)). By construction, the set A := ∞ n=1 An, Nn is n totally bounded and hence relatively compact. $n \rightarrow \infty$ n If $x * = -\infty$, then P[X* > $-\infty$] = lim P[X* > -\infty] > -n] = 0. For all $x \in E N$, $;;; ; ; ; ; n, k(x) - vn, k(x); \le Rn, k := k(k - 1)$. Now let (µn) $n \in N$ be a sequence in F. We assume that P[N = m] = 1 and show that this leads to a contradiction. (iii) Let X and Y be supermartingales and a, $b \ge 0$. It is easy to check that $d(\omega, \omega) := \infty 2 - i = 1$ di ($\omega(i), \omega(i)$) $1 + di (\omega(i), \omega(i))$ (14.2) is a complete metric on Ω that induces τ . Since ϕ and ϕ n are CFPs, $|\phi|^2$ and $|\phi_n|^2$ are also CFPs. Thus, since $|\phi_n(t)|^2$ pointwise, Lévy's continuity theorem implies uniform convergence on compact sets. , N and $n \ge n0$. Clearly, any map Xn : $\Omega \to R$, $\omega \to n - 1$, if $\omega n = 1$, ∞ , if $\omega n = 0$, 2 Warning: For some authors, the geometric distribution is shifted by one to the right; that is, it is a distribution on N., tn C t0, . s pt = That is, we have (d/dt)pt(x, y) = q pt(x, y). $\in E$ be the observed outcomes. Define pk = ak (mk+1 - mk) for k = 1, . If $D + \phi$ is continuous at x, then $D - \phi(x) = D + \phi(x)$. n n i=1 By Chebyshev's inequality (Theorem 5.11), for any $\varepsilon > 0$,) * V n $\rightarrow \infty$ P | Sn /n| $\geq \varepsilon
\leq 2 \rightarrow 0$. Exercise 13.3.1 Show that a family $F \subset Mf(R)$ is tight if and only if there exists a measurable map $f: R \to [0, \infty)$ such that $f(x) \to \infty$ for $|x| \to \infty$ and 3 sup $\in F f d\mu < \infty$. s $\leq t$ For $t < 1/2\lambda$, this implies rt = 0. to be uncorrelated. Proof Let $\gamma > 12$. Show the following: (i) d is a metric on the set of distribution functions. 21.2 The processes X n, n = 0, 1, 2, 3, 10 of the Lévy construction of Brownian motion. + Tn-1 Let $R := \{T1r + .$ Then, for every $x \in Zd$, $\mu(\{x\}) = (2\pi)-d$ [$-\pi,\pi$]d = -i)t, $x^* \phi \mu(t) dt$. We distinguish two cases: Case 1: t < n1. Particularly simple is the case where μ possesses an integrable density $f := d\mu d\lambda$ with respect to ddimensional Lebesgue measure λ . The dual space V of V is defined by $V := \{F : V \text{ of } V$ \rightarrow R is continuous and linear}. Iterating the argument, for every k \in N and $0 \le t1 \le t2 \le .$ Furthermore, f 22 = b02 + ∞ (a n=1 n + bn). t \in R By (23.9) and the differentiable infinitely often and the first two derivatives are) * ϕ (t) = E X1 et X1 and) * ϕ (t) = E X1 et X1 . Let A \in F. For the multidimensional situation, there are various possibilities for degeneracy depending on the size of the kernel of C. As in Example 1.11(vi), we define the set of all sequences whose first n values are $\omega 1$, Lemma 15.12 Let)X be a* random variable with values in Rd and characteristic function ϕX (t) = E ei)t, X*. This Springer imprint is published by the registered company Springer Nature Switzerland AG. Since ϕ is convex, for $y \in I \circ such that <math>y > x$, we have $\phi(y) \ge \phi(x) + (ii)$ (iii) (iv) (v) (v) $(y - x \phi(x) - \phi(x - h) h$ for all h > 0 with $x - h \in I \circ .$ $n \rightarrow \infty$ The additional statement is trivial as $E^{\sim} := E \cup \{\Omega\}$ is a π -system that generates A, and the value $\mu(\Omega) = 1$ is given. (In fact, using Exercise 21.4.2, it can even be shown that (Wt)t ≥ 0 converges to W almost surely and in L1 .) & Exercise 17.3.2 Let r, s, R, S \in N. Theorem 16.6 Let (ϕ n)n \in N0 , after 2n steps each path splits into three (see Fig. Although it might seem that these two events are entangled in some way, they are stochastically independent. Example 23.4 If PX1 = $\log 4$ is well-defined if $f \in Lp(\mu)$ and if μ is finite but it need not be if μ is infinite. Hence, by (7.8), also A (1 - g) $d(\mu + \nu) = 0$. Hence ϕ (t) = $\exp(\mu(Al)(eitl-1))$. k=i By assumption and using Fubini's theorem, we get fl-1 (ω l-1 , $d\omega$ l+1) fl+1 (ω l-1 , $d\omega$ l+1) fl+1 (ω l+1) k]i + 1 (ω l+1) fl+1 (ω l+1) (A2) for all $A2 \in A2$., such that Y1, Y2, . Our graph (V, E) is the starting point for a stochastic model of a porous medium. n n Hence $\mu(Nn) = 0$ for any $n \in N$ and thus $\mu(N) = 0$ for any $n \in N$ and thus to the so-called weak* -topology. If h = 0, then F β does not have a minimum at m = 0. On the other hand, $1(0,1) : R \rightarrow R$ is lower semicontinuous.) An equivalent condition for lower semicontinuous but not continuous.) An equivalent condition for lower semicontinuous but not continuous.) An equivalent condition for lower semicontinuous but not continuous.) randomness. In this case, we give a representation theorem and use it to discuss the fair price for a European call option in the stock market model of Cox- Ross-Rubinstein. n=m Example 2.8 We throw a die again and again again and again again and again compute $P[Ns = k, Nt - Ns = l] = P[Tk \le s < Tk+1, Tk+l \le t < Tk+l+1] \propto monotone$. If E is locally compact, it is enough to the second bet X1, X2, Then $F\{i\}(x) = 0 \times \theta i = -\theta i \times \theta i =$ consider f with compact support., m} $(L = n) = P \{Mn, 1 = k1, . Klenke, Probability Theory, Universitext, 435, 436, 18 Convergence of Markov Chains eigenvalue of p. 1 - 2 - \gamma (21.8) 21.1 Continuous Versions 521 Define C0 = 21 + \gamma (1 - 2 - \gamma) - 1 < \infty$. Thus, for large n, the Boltzmann distribution is concentrated on those x that minimize the free energy. P[C] 252 11 Martingale Convergence Theorems and Their Applications Clearly, X is adapted to F. $\phi(t)$ 15.2 Characteristic Functions: Examples 337 338 15 Characteristic Functions: Examples 337 338 15 Characteristic Functions: Examples 337 338 15 Characteristic Functions and the Central Limit Theorem Proof (i) (Normal distribution) By Lemma 15.12, it is enough to consider the case $\mu = 0$ and $\sigma 2 = 1$. Numbers We show Etemadi's version [47] of the strong law of large numbers for identically distributed, pairwise independent random variables. Then each of the families (Yn (x))n \in N and (Zn (x \cup An) = n (-1)k-1 μ (Ai1 \cap . For x \in E N , let 1 n ξ n (x) = n i=1 δ xi \in M1 (E). We will show that for any two finite sets J and J with J \subset J \subset I , P + i \in J , if i \in J \setminus J. 7.4 Lebesgue's Decomposition Theorem 175 (iv) W = L2 ([0, 1], λ). As soon as we know the value of X, we toss n times a coin that has probability X for a success. Definition 14.4 (Product- σ -algebra) Let (Ω , Ai), i \in I, be measurable spaces. Compute the (0 3/4 1/4 invariant distribution and let (Xt) t ≥ 0 be an F-supermartingale with right continuous paths. However, all complex Nth roots of unity $e^{2\pi i k/N}$, k = 0, Thus, in order to show (23.11), it is enough to show lim inf $n \rightarrow \infty$ ' (1 $^{1} \log E e^{-\tau} Sn 1{S^{n} \ge 0} \ge 0$. If the occurrence of the event does not change when we change finitely many values of the random variables, then the event is called terminal. Hence the generated filtration is the smallest filtration to which the process is adapted. If $n - f p \rightarrow 0$, Lp then we say that (fn)n \in N converges to f in Lp (µ) and we write fn $\rightarrow f$. Hence) $\sqrt{\sqrt{*t} |X|} = P Y 2 \leq t (X2 + Y 2) \propto 12 = dx dy e^{-(x + y)/2} 1 \{y 2 \leq t (x 2 + Y 2)\}$. k=0 We conclude + RW = n $\propto n=0$ k=0 k $\geq n+n0-1$ n=0 k=0 k $\neq \infty e^{-(x + y)/2} 1 \{y 2 \leq t (x 2 + Y 2)\}$. $0 \text{ n} \rightarrow \infty$ Now, by Theorem 19.33, we get $Xn \rightarrow -\infty$ almost surely. For measurable f, $g: \Omega \rightarrow E$, let $\tilde{g} := d(f, \infty N=12-N1 + \mu(AN) - 1 \wedge d(f(\omega), g(\omega)) \mu(d\omega)$. > X(n) . Indeed, $\mu \omega \in \Omega + \mu(\Omega) = \infty$, but $\omega \in \Omega + \mu(AN) - 1 \wedge d(f(\omega), g(\omega)) \mu(d\omega)$. variation in terms of a coupling: Let $D := \{(x, x) : x \in E\}$ be the diagonal in $E \times E$. We conclude that $\rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 =
\varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$ ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon - 1$) ($\varepsilon \rho C_{\varepsilon} c d\mu 1 = \varepsilon$ $|X = x] f(x, y) = fY |X(x, y) := for PX [dx]-a.a. x \in R$. Mainz March 2013 Achim Klenke vii Preface to the First Edition This book is based on two four-hour courses on advanced probability theory that I have held in recent years at the universities of Cologne and Mainz. Then E[X + Y | Y] = E[X | Y] + E[Y | Y] = E[X] + Y., Y1D) has covariance matrix Ci, := E[Y1i · Y1] = D2 1{i=j} . 23.4 Weiss ferromagnet: magnetization mβ,h as a function of β. In this case, for the time being it suffices to get acquainted with the statements of the Portemanteau theorem (Theorem 13.16) and Prohorov's theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.16) and Prohorov's theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), as well as with the statements of the Portemanteau theorem (Theorem 13.26), 13.29). To put it differently, there must be a function Fn : $\{-1, 1\}n-1 \rightarrow N$ such that Hn = Fn (D1, . Then A1 and A2 are semirings but A1 $\cap A2 = \{\emptyset, \Omega, \{1\}\}$ is not. k=1 y: $|y/k| \leq L/N$ By the weak law of large numbers, we have lim infk $\rightarrow \infty$ every $\varepsilon > 0$. $n \rightarrow \infty$ n $n \rightarrow \infty$ n nsatisfies conditions (LDP 1) and (LDP 2) at least for unbounded intervals. 23.2 Large Deviations Principle 595 Often (LDP 1) and (LDP 2) are referred to as lower bound and upper bound. Second Exercise 11.2.7 (Conditional Borel-Cantelli lemma) Let (Fn)n \in NO be a filtration and things together, we infer infz \in F (d(x, z) + d(z, y)) = d(x, y). By definition, $\kappa n (yn-1, yn-1 + An) = \mu n (An)$. 2 (1 - z) (1 + z) 1 - z Therefore, z 1 1 z log(1 + z) + log(1 - z) = $\mu 2$. \bullet Example 8.27 (i) Let ($\Omega 1$, A1) and ($\Omega 2$, A2) be discrete measurable spaces and let (Kij) i $\in \Omega 1$. Theorem 18.9 Let X be a Markov chain on E with transition matrix p. Now assume that I \subset R is closed under addition., N. \clubsuit 8.2 Conditional Expectations Let X be a random variable that is uniformly distributed on [0, 1]., d - 1} such that $nd + Lx, y \in N(x, y)$ for all $n \ge nx, y$. M Choose a countable basis U of the topology consisting of relatively compact sets. (ii) This is similar to (i). 257 257 263 266 13 Convergence of Measures ... A In the following section, we will need the representation theorem for the space L2 (μ), which, unlike L2 (μ), is not a Hilbert space. Hence, let $l \ge 1$. Exercise 6.1.1 Let Ω be countable. Consider first the one-dimensional simple random walk that with probability 1 - p jumps one step to the left. Theorem) 1 2 * 2.45 For d = 1, we have pc = 1. However, if $\#\Omega = \infty$, then A is not a σ algebra. Note that each point $x \in TL$ has exactly three neighbors which are in UL. Ω Let Ω be an (at most) countable nonempty set and let A = 2. Then An = n E[|Xi| Fi-1] - |Xi-1|. In particular, an F-(sub-, super-) martingale with respect to its own filtration $\sigma(X)$. "(ii) \Rightarrow (iii)" This works as in the proof of A = 2. Then An = n E[|Xi| Fi-1] - |Xi-1|. Theorem 6.25. By Theorem 1.96, there are g1, g2, . Hint: Consider E = [0, 1]. Markov chains with discrete state spaces give rise to many interesting probabilistic examples. Then there exist an $\varepsilon > 0$ and sets An \in A with $\mu(An) < 2-n$ but $\nu(An) \ge \varepsilon$ for all $n \in N$. Definition 1.69 A measure space (Ω, A, μ) is called complete if $N\mu \subset A$. Define $\mu : A \rightarrow [0, 2-n]$. ∞) by μ (a, b] \cap Q = b - a. be i.i.d. random variables with continuous distributed and centered. Theorem 1.3 If A is closed under complements, then we have the equivalences A is $\sigma \cdot \cap$ -closed $\Leftarrow A$ is $\sigma \cdot \cap$ -closed. Letting $\epsilon \downarrow 0$, we get C = Cb (E; R). Hence, by the residue theorem for $0 < b < c < \infty$, c x b r - 1 exp(-z) dz = 2r - 1 exp(-z) dz + zr - 1 exp(-z) dz. For any $\pi \in \Pi_0, \pi$, the probability that π uses only $v < c < \infty$, $c x b r - 1 exp(-z) dz = 2n - 1 \sqrt{e^{-x}/2} dx = 2n - 1 \sqrt{e^{-x}/2} dx$ open edges is $P[\pi \text{ is open}] = pm$., Uzn of E consisting of such neighborhoods and define $gx = \min(hz_1, ..., hence)$, we have $P[Xt > n] = P[Z_1 \le t] = P[Z_1 \le t]$ uniqueness for measures on σ (A) that were first defined on a semiring A only. Theorem 15.25 (Pólya) Let $f: R \to [0, 1]$ be continuous and even with f(0) = 1. are i.i.d. random maps $E \to E$ with P[F(x) = y] = p(x, y) for all $x, y \in E$. Theorem 21.37
Weak convergence in M1 (Ω , d) implies fdd-convergence: $n \to \infty$ Pn $- \to P = n \to \infty$ Pn $- \to$ Theorem In the following, let E be a Polish space with Borel σ -algebra E. Definition 19.34 The process X is called a random walk in the random environment W. Proof This is left as an exercise! Example 9.17 Let I = N0 (or, more generally, let I \subset [0, ∞) be right-discrete; that is, t < inf I \cap (t, ∞) for all t \geq 0, and hence I in particular is countable) and let $K \subset R$ be measurable. 7 Lp -Spaces and the Radon-Nikodym Theorem 186 3 In other words, $(f + \varepsilon 1A) \in G$ and thus $(f + \varepsilon 1A) = \gamma + \varepsilon \mu(A) > \gamma$, contradicting the definition of γ . 8.1 Elementary Conditional Probabilities .. Show that G(x, y) = G(x, y) for all $x, y \in E \setminus A$ and that $G(x, y) = 1\{x=y\}$ if $x \in A$. Show that $dP(P, Q) \leq dW(P, Q)$ for all P, Q for all Y. $Q \in M1$ (E). 10.2 Optional Sampling and Optional Stopping 235 Now { $\sigma < n$ } $A \in F\sigma$ An for $A \in F\sigma$. It measures the amount of new randomness added in each step. Let P0 = $\delta 0$ and let P be the unique probability measure on Ω corresponding to P0 and (xt : t ≥ 0) according to Corollary 14.47. Such a process is said to be binary splitting. $\leq f(Yn) \leq f(Yn) \leq 10^{-10}$ n n k=0 Hence E[f (Yn)] \leq 1 n n-1 k=0 E[f (|Xk|p)] = C. In both cases, we infer that bounded harmonic functions are constant. We conclude that lim sup Xm - Xn ∞ = 0 P-almost surely. Hence we have to postulate λ = 0. This justifies the name "exchangeable event". Hence (see Example 12.28) **)) E Z n = E P X1 = \cdots = Xn = 1 Z = P [Sn = C In both cases, we infer that bounded harmonic functions are constant. n = (N - 1)! (M + n - 1)! (M + n - 1)! (M + n - 1)! for all $n \in N$. A function $f : E \to F$ is called Lipschitz constant, with dF (f (x), f (y)) $\leq K \cdot dE$ (x, y) for all x, $y \in E$. A constantly updated list of errors can be found at www.aklenke.de. * Exercise 15.3.4 (Continuity theorem for Laplace 15.3.4) transforms) Let $(\mu n) \in N$ be a sequence of probability measures on $[0, \infty)$ and let ψn $(t) = e - t x \mu n$ (dx) for $t \ge 0$, $n \in N$ be the Laplace transforms. Takeaways Let us imagine the edges of some graph as resistors in an electrical network. First consider a general (discrete time) irreducible random walk with transition matrix p on ZD. N+1 Since N is consider a general (discrete time) irreducible random walk with transition matrix p on ZD. odd, $|\lambda k|$ is maximal (except for k = 0) for k = N-12 and for k = 2. Then: 3 3 (i) (Monotonicity) If $f \le g$, then $f d\mu \le g d\mu$. Show that the following two statements are equivalent: (i) There is a Borel measurable map $g: \Omega \rightarrow \mathbb{R}$ with f = g everywhere. If we define X = Z1 + Z2 and Y = Z1, then $(X, Y) \sim N\mu, \Sigma$ is bivariate $2\sigma 1 + \sigma 22\sigma 12$ normally distributed with covariance matrix Σ := and with μ := $\sigma 12 \sigma 12 \mu 1 + \mu 2$. As typically the option is exercised only if the market price at time T is larger than K (and then gives a profit of XT - K as the stock could be sold at price XT on the market), the value of the option is indeed VT = (XT - K)+ . , Xn be i.i.d. exponentially distributed random variables with parameter 1. $n \rightarrow \infty$ Hence, f is uniformly continuous. This is the case in particular if [0, 1] is replaced by $G \subset Rd$ for very large d. $n \rightarrow \infty$ Letting $\epsilon \downarrow 0$ yields d^{*}N (f, fn) $- \rightarrow 0$. Since $A \cap B = A$, we obtain $\mu(B) = \mu(A | B \setminus A)$ if $B \setminus A \in A$. Thus let $N = \{f \text{ is measurable and } f = 0 \ \mu - a.e.\}$. Then (aX + bY) is a supermartingale Now let Sn := T1 + . Hence $\mu(E) = \infty = \nu(E)$ for all $E \in E$. Define $A0 = \xi0$ and $\sqrt{2} \xi n$ An $:= \pi n$ for $n \in N$. Definition 24.3 A random measure on E is a random variable X on some and with $P[X \in M(E)] = Q$ probability space (Ω, A, P) with values in (M(E), M) 1. Proof For fixed N > 0 and s, t $\in [0, N]$, we have) *)) * * Ex (Yt + s - Ys) 4 = Ex EYs [(Yt - Y0 + S(X)4] = Ex 24Ys t 3 + 12Ys2 t 2 = 24x t 3 + 12(2sx + x 2) t 2 ≤ 48Nx + 12x 2 t 2. For t = 0, the claim holds trivially. t By the dominating function 1 \wedge x for t ≥ 1, we infer t $\rightarrow \infty$ t -1 (1 -e-tx) $\nu(dx) \rightarrow 0$. Show that any countable (respectively finite) union of sets in A can be written as a countable (respectively finite) t $\rightarrow \infty$ t -1 (1 -e-tx) $\nu(dx) \rightarrow 0$. Show that any countable (respectively finite) union of sets in A can be written as a countable (respectively finite) t $\rightarrow \infty$ t -1 (1 -e-tx) $\nu(dx) \rightarrow 0$. Show that any countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (respectively finite) union of sets in A can be written as a countable (finite) disjoint union of sets in A. {-n,-n+1,...} P Then {-n, -n + 1, ... = σ 2 . 470 19 Markov Chains and Electrical Networks and so on, yielding u(k) - u(0) = I (x1) k-1 R(l, l + 1). A Chapter 6 Convergence and conver in probability of random variables. Its row sum is denoted by $Sn = Xn, 1 + . + Xn n \rightarrow \infty \sqrt{\Rightarrow} N0, 1$. P Xn = 1 X1, X2, . (i) $\leftarrow \Rightarrow$ (iii) $\leftarrow \Rightarrow$ (iv) This follows from the elementary theory of power series. Let $\mu 1 = \pi, \mu 2$, . Even after a long time, the computer simulation does not produce the equilibrium state but rather so-called metastable states, in which the Weiss domains are clearly visible. Use the star-triangle transformation to remove the lower right node (left $1 = \delta/R1 = 19/6$, in Fig. Hence X is positive recurrent if and only if $M < \infty$. 254 11 Martingale Convergence Theorems and Their Applications ∞ Hint: Apply Exercise 11.2.5 to Xn = n=1 (1An - P[An |Fn-1]). n n-1 Yn \leq k=0 500 20 Ergodic Theory Again, by Jensen's inequality (now applied to f), we get that n-1 n-1 1 p |Xk | f (|Xk | p). A signed measure can be written as the difference of two finite mutually singular measures (Jordan decomposition). We show that PSn /n n \in N satisfies an LDP with rate n and rate function $\Lambda *$. The map Fn : R \rightarrow [0, 1], x \rightarrow n1 1($-\infty,x$] (Xi) is called the empirical i=1 distribution function of X1, . 7.5 Supplement: Signed Measures 179 Let X1 , . * Exercise 2.1.2 Show that the conclusion of the interesting part of the Borel-Cantelli lemma (Theorem 2.7(ii)) still holds under the following weaker condition: Each of the families (A1, A3, A5, . Use the Cauchy-Schwarz inequality for X and $1\{X>0\}$ in order to show the PaleyZygmund inequality P[X > 0] \geq E[X]2. For more general measurable functions, the integral was then defined as the limit of integrals of approximating elementary functions. Lemma 23.9 Let $N \in Then N$ and let as i, i lim sup $\varepsilon \log \varepsilon \rightarrow 0 N = 1$, $\bullet 1.5$ Random Variables The fundamental idea of modern probability theory is to model one or more random variables The fundamental idea of modern probability theory is to model one or more random variables. Lemma 23.9 Let $N \in Then N$ and let as i, i lim sup $\varepsilon \log \varepsilon \rightarrow 0 N = 1$, $\bullet 1.5$ Random Variables The fundamental idea of modern probability theory is to model one or more random variables. large numbers yields I:n \rightarrow I a.s. Note that the last theorem made no statement on the speed of convergence. $\in \mathbb{R}$ and $\alpha n \ge 0$ for all $n \in \mathbb{N}$ such that pn(y) d+i(y)(x0, y) > 0; hence $y \in Ei(y) \cap E$ i for every $i = 0, . "(ii) \Rightarrow (i)$ " Let $(\mu n) n \in \mathbb{N}$ be tight and let $\mathbb{C} \subset \mathbb{C}$ (E) be a separating class with (13.9). $n \in \mathbb{N}$ Show that $C = A + = A - = F \pmod{P}$. 1 However, this is the L (P)-convergence in (8.7). Then $\kappa(x, \cdot) = P[X + x \in \cdot] = \delta x * \mu$ defines a stochastic kernel from Rn to Rn . If, in addition, E is Polish, then (M(E), τv) is again Polish (see, e.g., [82, Section 15.7]). Now X jumps from n to n + 1 at rate n. Let $P[T \in N0] = 1$ and let X1 , X2 , . By Exercise $\infty 6.1.4$ since (iii) (Y
- E[Y]) converges a.s. As (ii) holds, holds, the series ∞ n n =1 n=1 Yn converges almost surely. In most cases, the events of Ω are not observed directly. Definition 3.9 (Zn)n $\in \mathbb{N}^{2}$ (Zn)n $\in \mathbb{N}^{2}$ (Xn $\rightarrow \Omega, x \rightarrow x + r \pmod{n}$. Furthermore, P[Ym > k] = $P[Xm, l = 0, l = 1, . In general, the value of pc is not known and is extremely hard to determine. Then the following are equivalent: n \rightarrow \infty$ (i) Pn $- \rightarrow P$ and (Pn)n $\in N$ is tight. In particular, I would like to thank Philipp Neumann for many helpful comments. (23.17) 604 23 Large Deviations Then lim $\epsilon \log \epsilon \rightarrow 0 e\phi(x)/\epsilon \mu\epsilon$ (dx) = sup $\phi(x) - I(x)$. Let KL be the set of all edges which have at least one endpoint in BL . + Xn = k] zk k=0 n=0 = ∞ P[T = n] ψ X1 (z) . be independent real random variables in L1 (P). For the general case, see, e.g., [71, page 293, Theorem 33.3]. For every n \in N, nY is distributed as the sum of n2 independent, standard normally distributed D random variables nY = Y1 + . Assume that any $n \in N$, the partition t n+1 is a refinement of t n; that is, for n+1 n + 1 n = 1 $p[X = k] = \infty$ lim supn $\rightarrow \infty E[X \ k \ by$ Hadamard's criterion $\psi X(z) := k=1$ $P[X = k] = \infty$ lim inf $k \rightarrow \infty$ Hence $f \in L1(\mu)$. Ergodic theorems are laws of large numbers for $(Sn) n \in N$. Proof We face the problem that the space $[0, \infty)$ is not compact by passing to the one-point compactification $E = [0, \infty]$. (i). Remark 1.24 Any of the classes E1, E2, E3, E5, The left figure shows the actual edges where, e.g., xyy indicates that the first step is in direction x, the second step is in direction y and then the third step is necessarily also in direction y. Then X is also adapted to F; however, in general, F σ (X), T }) – K bT, p* ({A, (iii) The family (δn) $n \in N$ of probability measures on R is not tight. Reflection Why do we need condition (iii) in the definition of the Poisson process? The sequence (μn) in X converges weakly to $\mu \in X$, if $n \to \infty \Phi(\mu n) - \to \Phi(\mu)$ for every $\Phi \in X$. \bullet Theorem 21.50 We have Lx [Zⁿ] \rightarrow Lx [Y]. As reference chain, we choose a chain with transition probabilities q(x, y) = if y = x i for some i $\in \Lambda$, 1 # Λ , 0, else. For $-\infty \leq a < b \leq +\infty$, we have lim P[Sn* $\in [a, b]$] = $\sqrt{12\pi}$ 3b e-x 2/2 dx. be independent random variables. = (-4) n n Takeaways Generating functions determine a probability distribution on N0. l=1 Concluding, we get P[Ym = k] = P[Ym > k - 1] - P[Ym > k] = p(1 - p)k. 15.3 Lévy's Continuity Theorem 345 As F is tight, there exists an N \in N with $\mu([-N, N]d) > 1 - \epsilon^2/6$ for all $\mu \in F$. For n = 1, . Then Xi ~ bn,pi and X1 \leq X2 almost surely. Then there exists X- ∞ = lim X-n almost surely and in L1. Hence the statement follows from Theorem 2.13. How must we choose the sequence (pn) $\in \mathbb{N}$ and $t \in \mathbb{R}$, we have kn 2 $1 - \phi_n$, $(t) \leq t$. A Exercise 4.2.6 Let λ be the Lebesgue measure on \mathbb{R} , $p \in [1, \infty)$ and let $f \in Lp(\lambda)$. Let $f \in Lp(\lambda)$ and $g \in Lq(\mu)$ be nontrivial. + Tn-L n + 1 s r r T + . n $\rightarrow \infty$ (ii) $\mu = v$ -lim μn (E). The probability measure P(Xj)j \in J on RJ is called the joint distribution of (Xj)j \in J on RJ is called the joint distribution functions or R. .} is an independent family of random variables. $n \rightarrow \infty$ Clearly, the sequence ($\delta 1/n$) $n \in N$ of probability measures on R converges weakly to $\delta 0$; however, not in total variation norm. While the method of the proof of Etemadi's theorem, we first present (and prove) a strong law of large numbers under stronger assumptions. Let Y = M + A be the Doob decomposition of Y. We do not strive for completeness but show only a few of the statements. Hence, by Remark 8.26, $x \rightarrow \kappa(x, A)$ is measurable for all $A \in B(Rn)$. Applying the monotone convergence theorem once more, we get $E[f(X) \ 1B] = \lim E[gn(X) \ 1B] = n \rightarrow \infty B f(x) \ \kappa X, F(\omega, dx) \ P[d\omega]$. In this case, $N := \{v \in V : v, v^* = 0\}$ and $V0 = V/N := \{f + N : f \in V\}$. A $i \in J$ is a more of the observation of the velocity of the notion of the velocity of the notion If d = 1, then both statements are equivalent. In particular, F (x, x) is the return probability (after the first jump) from x to x. Hence M is a bounded F-martingale and thus converges almost surely and in L1 to a random variable M ∞ . The claim follows by the uniqueness theorem for harmonic functions (Theorem 19.7). Then $\mu 1 \le st \mu 2$ if and only if there is a coupling ϕ of μ 1 and μ 2 with $\phi(L) = 1$. \in L2 (P) be pairwise independent (that is, Xi and Xj are independent for all i, $j \in N$ with i = j) and identically distributed. \blacklozenge Theorem 21.11 Let X = (Xt) t \in [0, ∞) be a stochastic process. \clubsuit) * Exercise 23.1.2 Let X be a real random variable and let $\Lambda(t) := \log E$ et X, $t \in R$ be its logarithmic moment generating function. (17.23) Now consider symmetric simple random walk. Let F be the filtration $F = (Fn)n\in N0$, where $Fn = \sigma$ (Ik, Nk: $k \le n$) for all $n \in N0$. $n \to \infty$ In order for $\mu pn \to \pi$ to hold for every $\mu \in M1$ (E), a certain contraction property of p is necessary. Hence ∞ An $\cap E = n=1 \infty$ (An $\cap E = n=1 \infty$ (An $\cap E = n=1 \infty$ (An $\cap E = n=1 \infty$). false since the supremum depends on more than countably many coordinates., kn, $n \in N$ be an array of CFPs with the property sup lim sup L>0 $n \rightarrow \infty$ sup sup t $\in [-L,L]$ l=1,...,kn $|\phi n, l(t) - 1| = 0$. Let k = Lemma 21.45) (t + s)n!. \Rightarrow Exercise 4.2.4 Let λ be the Lebesgue measure on R and let A be a Borel set with $\lambda(A) < \infty$., in , i \in E with P[Xs1 = i1, ...,kn | $\phi n, l(t) - 1| = 0$. Let k = Lemma 21.45) (t + s)n!. \in A with An $\uparrow \Omega$ and μ (An) < ∞ for any n \in N. Theorem 13.18 (Slutzky's theorem) Let X, X1 , X2 , . 18.6 Ising model (150 × 150 grid) below the critical temperature. Exercise 20.1.1 Let G be a finite group of measure-preserving measurable maps on (Ω , A, P) and let A0 := {A \in A : g(A) = A for all g \in G}. A $\subset 2\Omega$ is a σ -algebra A \supset E σ (E) is called the σ algebra generated by E. 3 fZ dµ = The equivalence of (ii) and (iii) was established in the preceding theorem. Hence we only show (iii) \Rightarrow (i). Furthermore, H ($\nu | \mu \rangle = 0$ if and only if $\nu = \mu$., n} $\rightarrow 0$. We check that this is indeed the case. n \in N $r, s \in Q$ Ar, s $\cup r \in Q$ Br $\cup C$. \clubsuit 15.3 Lévy's Continuity Theorem The main statement of this section is Lévy's continuity theorem (Theorem 15.24). If $\nu = f \mu$, then $\nu(A) = 0$ for all $A \in A$ with $\mu(A) = 0$., tN since these functions determine the distribution of $(Bt + \tau)t \ge 0$. Then An $(\varphi k + \pi)t \ge 0$. Then An $(\varphi k + \pi)t \ge 0$. Then An $(\varphi k + \pi)t \ge 0$. Then An $(\varphi k + \pi)t \ge 0$. Then An $(\varphi k + \pi)t \ge 0$. are independent, we have ∞ GY = 0 P0 [Yt1 = 0]D dt. Definition 1.76 (Measurable maps) (i) A map X : $\Omega \rightarrow \Omega$ is called A - A -measurable (or, briefly, measurable) if X-1 (A) := {X-1 (A) : A \in A } C A; that is, if X-1 (A) := {X-1 (A) : A \in A } C A; that is, if X-1 (A) : A \in A In order to derive the formula for u(x), we make the following observations. Hence the space of all possible outcomes of the repeated experiment is $\Omega = E N$. Since X is irreducible and aperiodic, by Lemma 18.3(ii), there exists an $N \in N$, such that the N-step transition matrix fulfills pN (0, x) > 0 for all $x \in \{-1, 0, 1\}d$. We assume that (iii) does not hold and produce a contradiction. For example, we have \times n [ai, bi) = i=1 $\propto \times$ n k=1 i=1 1 ai -, bi $\in \sigma$ (E5). If ϕ is twice continuously differentiable, then ϕ is convex if and only if the Hessian matrix is positive semidefinite. Independent random variables are uncorrelated. We write X $\sim \mu$ if $\mu = PX$ and say that X has distribution μ . Birkhoff's ergodic theorem now implies that, for every $x \in E$, $1 n \rightarrow \infty 1\{Xk = x\} \rightarrow \pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s.
n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. n n-1 k=0 In this sense, $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a.s. $\pi(\{x\}) P\pi$ -a. for any $N \in N$, there is a $\mu N \in F$ with μN (Acn, $N \ge \delta/2$. \blacklozenge Theorem 14.19 (Fubini) Let (Ωi , Ai, μi) be σ -finite measure spaces, i = 1, 2. (N - 1 + n)! (M - 1)! (N - M - 1)! The right-hand side depends on sn only and not on the order of the x1, . See Fig. Then μ is uniquely determined by the values $\mu(A)$ for all $A \in Z E$, R. Let $J \subset N$ be finite and n := 1, 2. (N - 1 + n)! (M - 1)! (N - M - 1)! The right-hand side depends on sn only and not on the order of the x1, . max J. & Chapter 13 Convergence of Measures One focus of probability theory is distributions that are the result of an interplay of a large number of random impacts. are i.i.d. with distribution function.) Then $\Gamma \theta$, r is called the Gamma distribution with scale parameter θ and shape parameter r. $2\pi - \infty - \infty P[\zeta \le t] = P 1 - t |Y| \le Passing$ to polar coordinates, we obtain $P[\zeta \le t] = 1 2\pi \infty 0 r$ dre-r 2/2 $2\pi 0 d\phi 1 \{ sin(\phi)2 \le t \} = \sqrt{2} arc sin t$. Then, for any $A \in A$ with P[A] > 0 and any $k \in I$, P[Bk|A] = P[A|Bk] P[Bk]. #A i i c A life we consider a very large system, then we are close to the so-called thermodynamic limit $m(\beta) := \lim m\Lambda(\beta)$, $\mu n \cdot Let \mu 1$, $\mu 2 \in Mf(E)$ be such that $f d\mu 1 = f d\mu 2$ for all $f \in Lip 1$ (E; [0, 1]). Let $m-1 X := \xi_{0,1} B_{0,1} + n n 2 \xi_{m,k} B_{m,k}$, m=1 k=1 and define $X^{\sim} t$ as the L2 (P)-limit $X^{\sim} t = L2 - \lim n \rightarrow \infty Xtn$. Hence we get the bound $\#TL \leq \#(BL \setminus BL-1)$ and thus $\#(BL \setminus BL-1) dL \rightarrow \infty \#TL \leq d = 1$ 0. Therefore, it is enough to consider the case $\mu(E) \in (0, \infty)$. A point h with degHL (h) = 1 is called a leaf of HL. As the only exception, the systematic construction of independent random variables is deferred to Chap. This model is rather simple and describes an idealized market (no transaction costs, fractional numbers of stocks tradeable and so on) What does the result imply for the convergence of PZn?, $Xtn-2 \in An-2$, $Xtn-1 \in dxn-1 = An-1 \times tn -tn-1$ (Xtn-1, An); hence Px [$Xtn \in An | Ftn-1] = \infty$, $\mu(C)$ We show that the following three statements are equivalent. Without loss of generality, assume E0 \cap E i cyclically until this holds). That theorem thus makes a statement about the length of a binary prefix code needed to transmit a long message. 4.1 For the Riemann integral, the area under the curve is approximated by rectangles of a fixed breadth (left hand side). Then $E[X2n] = (-1)n \phi(2n)(0) < \infty$. Then there exists a unique decomposition X = M + A, where A is predictable with A0 = 0 and M is a martingale. By Theorem 7.26, there is an $f + N \in V0$ with F0(x + N) = x + N, $f + N \approx 0$ for all $x + N \in V0$. In the following we will assume for the stochastic integral that such a continuous version is chosen. Theorem 2.42 The map $[0, 1] \rightarrow [0, 1]$, $p \rightarrow \theta$ (p) is monotone increasing. 19.4 Recurrence and Transience We consider the situation where E is countable and A1 = {x1 } for some x1 \in E. Proof Let x0 \in I and let (xn)n \in N and such that lim xn = x0 is countable and A1 = {x1 } for some x1 \in E. holds: $(1 + x)\alpha = \infty$ $\alpha k = 0 k xk$ for all $x \in C$ with |x| < 1. Theorem 19.19 (Rayleigh's monotonicity principle) Let (E, C) and (E, C) be electrical networks with $C(x, y) \ge C$ (x, y) for all $x, y \in E$. Examples of Polish spaces are countable discrete spaces (however, not Q with the usual topology), the Euclidean spaces Rn , and the space C([0, 1]) of continuous functions $[0, 1] \rightarrow R$, equipped with the supremum norm $\cdot \infty$. The points in UL can be isolated (that is, without neighbors) or can be joined to arbitrarily many points in UL in the sense of weak convergence in M1 (C([0, ∞))), the rescaled Galton-Watson processes Z⁻ n converge to Feller's diffusion Y : Lx [Z⁻ n] \rightarrow Lx [Y]. However, there is a much stronger statement that here we can only quote (see [95], and see [111] for a modern proof). n-1 (12.6) For n \geq 2, give a nontrivial example for equality in (12.6). (16.1) In particular, $CPoi\mu + \nu = CPoi\mu + CPoi\nu$; hence $CPoi\nu$; hence $CPoi\nu$; hence $CPoi\nu$; hence (5.13) as above with c > 0. Proof This is trivial. For a similar model, the Weiss ferromagnet, we will prove in Example 23.20 the existence of such a phase transition. By Theorem 1.18, it is enough to show that $\delta(E)$ is a π -system. We carry out the proof of (2.8) by induction on #J. Let Sn = X1 + . Due to (13.10), we have $f d\mu - f d\nu \ge \varepsilon$; hence $\mu = \nu$. (ii) If $t * := \sup(I) \in I$, then $E[\phi(Xt *) +] < \infty$ implies (9.1). A more exhaustive investigation can be found in Spitzer's book [159]. n; hence P + n , + + , n = E $\ge \infty$ (Ai). After that, there are seven more or less independent parts, consisting of Chaps. (13.10) 13.3 Prohorov's Theorem 295 By Prohorov's theorem, there exists a $\nu \in M \le 1$ (E) and asubsequence 3 3 (nk) k N of (nk) k N with μ N $\rightarrow \nu$ weakly. Hence Lemma 4.6(i) implies $\mu(A) = 1$ A d $\mu \leq 1$ n f 1{f $\geq n$ } d $\mu \leq 1$ n f 1{f = n} d $\mu \leq 1$ n f 1{f $\geq n$ } d $\mu \leq 1$ n f 1{f = n} Find an example of a pointwise convergent sequence of characteristic functions (ϕ n) such that the limiting function ϕ is not continuous (at 0). Let X be a martingale. Let E0 :=)(x1, . Let P[T \in N0] = 1 and assume that X1, X2, . Hence, it is enough to show for every $\delta > 0$ that lim sup l(kn)-1 max{|Sk| : k ≤ kn} ≤ δ almost surely. Consider the torus $E := \operatorname{Rd}/(2KZd)$ and define $f^{\circ}: E \to \operatorname{R}$ by $f^{\circ}x + 2KZd = f(x)$ for $x \in [-K, K)d$. s, p Exercise 3.1.2 Give an example for two different probability generating functions that coincide at countably
many points $x \in (0, 1)$, $i \in \mathbb{N}$. For any $\omega 1$, . Then the map $Y := X \circ X : \Omega \to \Omega$, $\omega \to X(X(\omega))$ is A - A -measurable. Denote the conductances of these wires by C1,., Xn (ω) occurs., 1} with transition matrix $\begin{bmatrix} x(1-x), \text{ if } y = x + 1/N, \\ y = x + 1/N$ $\mu(Bn) + n = 1 = \infty$ $\mu(Cnk)$ mn $\mu(Cnk) k = 1$ ($n = 1 \le \mu * (E) + \epsilon$. 218 9 Martingales Proof Let A be measurable and $t \in I$. $\in D$ are pairwise disjoint and $A := \infty n = 1$ An, then I1A = ∞ I is measurable; hence $A \in D$. Show that y is also positive recurrent. More precisely, if we let $\epsilon 0 := (1 - (2r - 1)2) \sin(2\pi/N)2 + (1 - (2r - 1)2) \sin(2\pi/N)2 + 2$ $\cos(2\pi/N)$ then the eigenvalue with the second largest modulus has modulus $\gamma \varepsilon = |\lambda \varepsilon, N/2| = 1 - 2\varepsilon$, if $\varepsilon \le \varepsilon 0$, or $\gamma \varepsilon = |\lambda \varepsilon, 1| K = 2$ $2\pi = (1 - \varepsilon) \cos 2\pi + \varepsilon + (1 - \varepsilon)(2r - 1) \sin \varepsilon$, N N if $\varepsilon \ge \varepsilon 0$. If, more generally, B is measurable with $\mu(B) = 0$, then $\mu(B \setminus E) = 0$; hence, as shown above, $\nu a(B) = \nu a(B \setminus E) = 0$. Let $p \in (1, \infty)$, $f \in Lp$ (λ) , where λ is the Lebesgue measure on R. Choose K < ∞ large enough that $\{h \ge K\}$ h d $\mu < \delta/2$. (iii) In general, $\phi \in M \pm$ is not σ -subadditive. Corollary 6.22 If (Xi)i \in I is a family of square integrable random variables with sup $\{|E[Xi]| : i \in I\} < \infty$ then (Xi)i $\in I$ is uniformly integrable. $k \rightarrow \infty$ Then $gk \rightarrow 0$ almost everywhere and $g - gk \ge 0$. Clearly, indistinguishable processes are modifications of each other. For example, if σ (X) ⊂ F (that is, if we know X already), then E[X |F] = X, as shown in (iii). Proof (i) This has been shown already in Example 20.17. Recall the notation x ≤ y if x i ≤ y i for all i = 1, . , n, ◆ This example could still be treated by elementary means. ◆ Lemma 9.21 If σ and τ are stopping times with $\sigma \leq \tau$, then $F\sigma \subset F\tau$. Further, let $\Omega = R[0,\infty)$, $A = B \otimes [0,\infty)$ and let Xt be the coordinate map for $t \in [0,\infty)$. Denote by $C = \{u + iv : u, v \in R\}$ the field of complex numbers. Definition 17.1 We say that X has the Markov property (MP) if, for every $A \in B(E)$ and all s, $t \in I$ with $s \leq t$, **) P Xt $\in A$ Xs. (iii) If $d \geq 3$, then Px0 [$\tau x1 < \tau x0$ | $\tau x0 \land \tau x1 < \infty$] = 12. > yn = 0 are given. However, the definition of weak convergence of distribution functions is constructed so that no mass defect occurs in the limit. , ωn)) $\mu 1$ ($d\omega 1$, (17.14) y=x Finally we assume that (which is equivalent to exchangeability of the limit and the sum over y = x in the display preceding (17.13)) lim t $\downarrow 0$ 1 Px [Xt = y] - 1{x=y} = q(x, y) t for all x, y \in E. Any property of Brownian motion that can be checked arbitrarily early after time 0 has either probability 0 or 1. Show that X ~ Nµ,C., XT - 1, XT ±). On the other hand, we show that, for infinitely divisible μ , the sequence $\nu n = 1R \{0\}$ nµ*1/n does the trick. Proof Let $X * := \lim \inf Xn$. Star-triangle transformation (see Exercise 19.5.1). As we have shown already, the CRR model is complete. Define H0n = 0 and Htn = Hi2-n T (i + 1)2-n T for some i = 0, . We have studied both in this section. and Let $a \ge 0$. Such a decomposition $\Omega = \Omega - \Omega + is$ called a Hahn decomp respect to ϕ). Now let $f \in L1$ ($\mu 1 \otimes \mu 2$). In fact, every point is visited periodically after N steps. This is a special case of the situation where (V,) · , · *) is a linear space with complete semi-inner product. (ii) Let x1 , x2 , . (i) Show that (X (ii) Use Doob's inequality to show the following. n The remaining part of the proof is dedicated to verifying the verif reverse inequality: lim inf $n \rightarrow \infty$ 1 log P[Sn ≥ 0] $\geq \log .$, n) is independent. Hence the current flow I with respect to u satisfies $-I(A0) = I(x1, x) = C(x1) \setminus 3 = C(x1) + x \in A \subset E$ is called σ -compact if A is a countable union of compact sets. (21.12) is locally Hölder-continuous of order γ . Let μ and ν be measures on (Ω , A). Then Y := (f (Xn))n \in N0 is an integrable adapted process (since |f (Xn)| \leq maxx \in {x0 - n,...,x0 + n} |f (x)|). This also shows that the expectations on the left hand side never equal zero. \blacklozenge Theorem 14.25 Let (Ω i , Ai), i = 0, 1, 2, be measurable spaces. On the other hand, the Xt generate $B(\Omega, d)$. The domain of attraction $Dom(\mu) \subset M1$ (R) is the set of all distributions PX with the property that there exist sequences of numbers (an $n \to \infty \Rightarrow \mu$. By Example 15.16, every fn is a $n \to \infty$ characteristic function of a probability measure μn ., ik, we get '*) *) E f1 (X1) ··· fk (Xk) = E F (X1, . Hence the following limit exists (see Exercise 20.6.2) h := h(P, τ) := lim n $\rightarrow \infty$ 1 1 hn = inf hn. τ is a stopping time if and only if { $\tau = t$ } \in Ft for all t \in I. By Theorem 4.19, the canonical inclusion i : L2 (Ω , A, $\mu + \nu$) \rightarrow L1 (Ω , A, $\mu + \nu$) is continuous. }. By virtue of Theorem 2.13(i), this implies that the family (X1, . Then $(\mu \subset \mathbb{Z}: K\mu(\mathbb{C}) \leq \nu(\mathbb{C}) \setminus \mathbb{C}) = \mathcal{C} \in \mathbb{Z}: K\mu(\mathbb{C}) \leq \nu(\mathbb{C}) \mu(\mathbb{C}) \leq 1 \nu(\Omega) = \delta$; K 7.5 Supplement: Signed Measures 181 hence $(\nu(\mathbb{C}) = \nu \setminus \mathbb{C} \in \mathbb{Z}: K\mu(\mathbb{C}) \leq \nu(\mathbb{C}) \mu(\mathbb{C}) = \nu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) = \nu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) = \nu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) = \nu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) \mu(\mathbb{C}) = \nu(\mathbb{C}) \mu(\mathbb{C}) \mu($ distributions is in the domain of attraction of a stable distribution and for which parameter? As the sequence X1, X2, By (17.2), the finite-dimensional distributions of X are uniquely determined. Takeaways Consider an arbitrary product of measurable spaces. C(x1) (19.12) In particular, x1 is recurrent \Rightarrow Ceff (x1 $\leftrightarrow \infty$) = 0 \Rightarrow Reff (x1 $\leftrightarrow \infty$) = ∞ . (One could choose D1 as the union of the sets of a finite covering of B1 with balls of radius d(B1, Ac1)/2. An event with this property is called invariant. Let $x \in U \cap [0, \infty)$ with I (x) < ∞ (if such an x exists). For two probability measures P and Q on (E, B(E)), denote by K(P, Q) \subset M1 (E × E) the set of all couplings of P and Q. Theorem 15.24 (Lévy's continuity theorem) Let P, P1, P2, Proof (i) P is upper semicontinuous and σ -subadditive; hence, by assumption, * P[A] = lim P n \rightarrow \infty \infty. Finally, define a stochastic process (Xn) n \in N by Xn := C \in Zn : P[C]>0 Q(C) 1C . (21.3) 21.1 Continuous Versions 519 Proof (i) It is enough to show that, for any T > 0, the process X on [0, T] has a modification XT that is locally Hölder-continuous of any order $\gamma \in (0, \beta/\alpha)$. 21.9 Pathwise Convergence of Branching Processes . Finally, let Y1, . The restriction that F is countably generated can also be dropped. \bullet (17.26) Example 17.56 Let (E,) be a Polish space. or its affiliates Table of contents : Front MatterPages i-xiv Basic Measure Theory (Achim Klenke)....Pages 1-51 Independence (Achim Klenke)....Pages 53-84 Generating Functions (Achim Klenke)....Pages 85-94 The Integral (Achim Klenke)....Pages 113-146 Convergence Theorems (Achim Klenke)....Pages 147-162 Lp-Spaces and the Radon-Nikodym Theorem (Achim Klenke)....Pages 163-189 Conditional Expectations (Achim Klenke)....Pages 213-228 Optional Sampling Theorems and Their Applications (Achim Klenke)....Pages 213-228 Optional Sampling Theorems (Achim Klenke).....Pages 213-228 Optional Sampling Theorems (Achim Klenke).... Klenke)....Pages 257-271 Convergence of Measures (Achim Klenke)....Pages 303-326 Characteristic Functions and the Central Limit Theorem (Achim Klenke)....Pages 327-366 Infinitely Divisible Distributions (Achim Klenke)....Pages 367-389 Markov Chains (Achim Klenke)....Pages 303-326 Characteristic Functions and the Central Limit Theorem (Achim Klenke)....Pages 327-366 Infinitely Divisible Distributions (Achim Klenke)....Pages 367-389 Markov Chains (Achim Klenke)....Pages 303-326 Characteristic Functions and the Central Limit Theorem (Achim Klenke)....Pages 303-326 Characteristic Functions (Achim Klenke)....Pages 303-326 Characteristic Functions and the Central Limit Theorem (Achim Klenke)....Pages 303-326 Characteristic Functions (Achim Klenke).....Pages 303-326 Characteristic Functions (Achim Klenke).....Pages 303-326 Characteristic Functi Klenke)....Pages 391-434 Convergence of Markov Chains (Achim Klenke)....Pages 435-459 Markov Chains and Electrical Networks (Achim Klenke)....Pages 435-459 Markov Chains (Achim K (Achim Klenke)....Pages 587-609 The Poisson Point Process (Achim Klenke)....Pages 665-690Back MatterPages 665-660Back MatterPages 665-690Back MatterPages 665-690Back MatterPages 665-660Back MatterPages 665-690Back MatterPages 665-660Back MatterPages 665-690Back MatterPages 665-690Back MatterPages 665-690Back MatterPages 665-690Back MatterPages 665-660Back MatterPages 665-690Back Matter
....Pages 665-690Back MatterPages 665-690Back MatterPages 665-690Back MatterPages 665-660Back MatterPages 665-690Back Matte Universitext Universitext Series Editors Sheldon Axler San Francisco State University of California, Berkeley Claude Sabbah École Polytechnique, CNRS, Université Paris-Saclay, Palaiseau Endre Süli University of Oxford Wojbor A. All other implications are incorrect in general. 10.2 Optional Stopping Lemma 10.10 Let I \subset R be countable, let T \in I and let τ be a stopping time with $\tau \leq T$. Then, for all $\omega^2 \in \Omega^2$, $A\omega^2 1 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\} \in A^2$, $A\omega^2 2 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\} \in A^2$, $A\omega^2 2 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\} \in A^2$, $A\omega^2 2 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\} \in A^2$, $A\omega^2 2 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\} \in A^2$, $A\omega^2 2 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\} \in A^2$, $A\omega^2 2 := \{\omega^2 \in \Omega^2 : (\omega^2 1, \omega^2) \in A\}$ $\in \Omega 1$: $(\omega 1, \omega^2) \in A \} \in A 1$, $f\omega^2 1$: $\Omega 2 \rightarrow R$, $\omega 2 \rightarrow f(\omega^2 1, \omega 2)$ is A2 -measurable, $f\omega^2 2$: $\Omega 1 \rightarrow R$, $\omega 1 \rightarrow f(\omega 1, \omega^2)$ is A1 -measurable. $n := Now let X m = 1 \ [0,\infty)$ (Ym). There are some really good lecture series on YT for measurable. $n := Now let X m = 1 \ [0,\infty)$ (Ym). There are some really good lecture series on YT for measurable. $n := Now let X m = 1 \ [0,\infty)$ (Ym). theorem for Laplace transforms (compare Exercise 15.3.4) yields that the weak limit ν° := w-lim ν° n (in M1 ([0, ∞)) exists and is uniquely determined by u., xd), (y1, . 298 13 Convergence of Measures Step 1 (Finite subadditivity of β). For every K > 0, we have $\sqrt{3 \times 10^{-10}}$ we have 4.6(i) and (ii), we infer $0 \le f d\mu \le (\infty \cdot 1N) d\mu = \lim n \to \infty n1N d\mu = 0$. Inductively, we get $r \ge qn$ for all $n \in N0$; that is, $r \ge q$. 1.1 Inclusions between classes of sets $A \subset 2\Omega$. Lindeberg-Feller theorem, we then get Sn := Xn,1 + . 18.1 Periodicity of Markov Chains . However, the statement can also be deduced from Kolmogorov's 0-1 law as limes superior and limes inferior are in the tail σ -algebra. Furthermore, if I \subset R is an interval (not necessarily open) and ϕ : I \rightarrow R is convex, then we still have L(ϕ) = \emptyset . 0.95 \cdot 0.02 + 0.1 $0.98 ext{ 117 On the other hand, the probability that a device that was not classified as defective is in fact defective is in fact defective is in fact defective is <math>P[B | Ac] = 1 \ 0.05 \cdot 0.02 = \approx 0.00113$. Proof Let $F = \sigma$ (B) be the filtration generated by B and let $\tau < \infty$ be an Fstopping time. We have to show that P[F] = 0. Show that the fraction of black balls converges almost surely to a random variable Z with a Beta distribution and determine the parameters. Finally, define the matrix p by p(x, y) = qy * x for $x, y \in N0$. $n \to \infty$ By Theorem 13.16, Fn $(x) = \mu n ((-\infty, x]) - \mu ((-\infty, x]) = F(x)$. By Exercise 11.1.1, for $\lambda > 0$, we have $) * \lambda P \sup \{|Xt| : t \in I \}$ $Q + \cap [0, N] > \lambda$) * = λ sup P sup{ $|Xt| : t \in I > \lambda : I \subset Q + \cap [0, N]$ finite $\leq 6 E[|X0|] + 4 E[|XN|]$. Here we present a stronger inequality that claims the same bound but now for the maximum over all partial sums until a fixed time. Define a map F : (0, 1] $\rightarrow (0, 1]$ by F (x) = (0, x1 x1 x2 x2 x3 x3 . Manifestly, 1 is the largest (absolute value of an) \otimes The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. This is a specialty of dimension one that makes it easy to check if the random walk is transient or recurrent. Caua * Caub = Caua+b for a, b > 0. Let P = ni=1 µi be the product measure of the µi (see Theorem 1.61). As one possible choice for the basis consists of cosine functions, this procedure is known as frequency decomposition of Brownian motion. For the case Ω = Rd, one could take the open balls with rational coordinates (compare Remark 1.24). $= \lim n \rightarrow \infty$ xn. By Corollary 21.41, it is enough to show that, for a sequence (Kn) $n \in N$ (chosen later), the families ($L[U^- Kn, n], n \in N$) and ($L[T^- Kn, n], n \in N$ and ($L[T^- Kn, n], n \in N$) and ($L[T^- Kn, n], n \in N$ and ($L[T^- Kn, n], n \in N$) and ($L[T^- Kn, n], n \in N$ and ($L[T^- Kn, n], n \in N$) and ($L[T^- Kn, n], n \in N$ and (L[Tk = 0. For a fixed set $B \in A$ with P[B] > 0, the conditional probability $P[\cdot|B]$ is a probability measure. Independencies is one of the theory's major tasks. -s + 2 In order to compute the iterated function, first consider general linear rational ab functions of the form f(x) = 0. ax+b. D (iii) Fn \rightarrow F. Proof By Ohm's rule and Kirchhoff's rule, u(x) - C(x, y) = 0. n=1 = 1 By construction, A is totally bounded. * (iii) If $r \in \{0\} \cup 12$, 1, then X is transient. A map $f: E \rightarrow R = [-\infty, \infty]$ is called lower semicontinuous if, for every $a \in R$, the level set f - 1 ($[-\infty, a]$) $\subset E$ is closed. Hence there exists a countable base U of the topology τ on E; that is, E a countable set U of open sets such that $A = U \in U$, U $\subset A$ U for any open $A \subset E$. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the locally compact Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the local Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the local Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the local Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the local Polish space E. Theorem 21.27 Let (kt)t ≥ 0 be a Feller semigroup on the local Polish space E. Theorem 21.27 Let (kt (ii) The normal distribution is infinitely divisible with Nm, $\sigma 2 = Nm/n, \sigma 2 / n$. If the measures have Lebesgue densities, then we obtain an explicit formula for the density of the convolution. The analogous statement holds for rings, σ -rings, algebras and λ systems. $n \rightarrow \infty$ (i) If μ , $\mu 1$, $\mu 2$, . Then S is a martingale by the preceding theorem, be i.i.d. random variables that are uniformly distributed on [0, 1]. Hence we have $n \rightarrow \infty$ lim sup
E[f(X)] - $\rightarrow \infty$, and hence by the monotone convergence theorem m $\rightarrow \infty$ 3 3 (Lemma 4.6(ii)) and by monotonicity, gn \leq fn (thus gn dµ \leq fn dµ), lim inf fn dµ = lim n $\rightarrow \infty$ fn dµ. C \in C with Step 2(σ -subadditivity of β). $\Rightarrow \infty$, and hence by the monotone convergence theorem m $\rightarrow \infty$ 3 3 (Lemma 4.6(ii)) and by monotonicity, gn \leq fn (thus gn dµ \leq lim inf n $\rightarrow \infty$ fn dµ. C \in C with Step 2(σ -subadditivity of β). R be measurable with $\mu 1$ (dt1) $\mu 2$ (dt2) a(t1, t2) 2 < ∞ . Fix Nn \in N such that $\mu E \setminus B1/n xin < n$. We close with an application to a model from mathematical finance. Definition 12.25 The random measure 1 $\equiv n := \delta X i n n i = 1$ is called the empirical distribution of X1, 228 9 Martingales Letting A := min{i $\in N0 : (1 + b)i(1 + a)T - ix0 > ix0 >$ K}, we get $\pi(VT) = Ep * [VT] = T + (+bT, p* ({i}) (1 + b)i (1 + a)T - ix0 - Ki = 0 = x0T$ i= A T (T * i * T - i i T - i (1 + b) (1 + a) - KbT, p* ({i}). The functions fn := (f - f1) 1N c and f := (f - f1 array is independent, centered and normed. Show the following version of Benford's law: For every $d \in \{1, . Then a stochastic kernel kt is defined by kt (x, A) := Px [Xt \in A] for all <math>x \in E$, $A \in B(E)$, $t \in I$. Let $3x 2 Sn^* := \sqrt{1 - \infty} e^{-t/2} dt$ be the distribution $2\pi n\sigma$ function of the standard normal distribution. Hint: Show that Y is a continuous Gaussian process with the correct covariance function. (iv) If I = R and $\Omega 0 = R$, then RR is the set of maps $R \rightarrow R$. Hint: Show this first for the normal distribution N0, ϵ , $\epsilon > 0$. be i.i.d. nonnegative random variables. (ii) μ is called singular to ν (symbolically $\mu \perp \nu$) if there exists an $A \in A$ such that $\mu(A) = 0$ and $\nu(\Omega \setminus A) = 0$. n=1 (ii) Let N, X1, X2, . 23.4 Varadhan's Lemma and Free Energy 605 As by assumption infM>0 FM = $-\infty$, it is enough to show that for any $k \in N$ we have) * E Y 2k-1 = 0 and) * a support of (x) κX , F (ω , dx). Use Theorem 15.32(i) to show that for any $k \in N$ we have) * E Y 2k-1 = 0 and) * a support of (x) κX , F (ω , dx). $(2k)! E Y 2k = k \cdot 15.2$ Characteristic Functions: Examples 339 However, $|zr-1 \exp(-z)| \leq (1 + t 2)(r-1)/2$ br $-1 \exp(-b)$ for $z \in \delta b, t$. Use Hölder's inequality to show that Λ is convex and is strictly convex in the interval where it is finite (if X is not almost surely constant). Assume that $f: \Omega 1 \times \Omega 2 \rightarrow R$ is measurable with respect to $A1 \otimes A2 \dots +Xn$. (6.8) Indeed, for $\varepsilon > 0$ and I = (ε, ∞), we have d sup f (x, λ) = sup x e - $\lambda x = \varepsilon - 1 e - 1 < \infty$. 1.3 The Measure Extension Theorem 29 The argument of Example 1.56 yields the following theorem. Then Nn \uparrow N and 0 = f d $\mu \ge \mu$ (Nn) 1 1N d $\mu = .3$ (ii) There is an f \in L1 (μ) with $\phi = f \mu$; hence A f d $\mu = \phi(A)$ for all A \in A. This allows for greater freedom in the 3 2 choice of ν than in the case of nonnegative random variables.) Now Var[X] = x νk (dx). Assume that $f: \Omega \rightarrow R$ is μ integrable. For s, $t \in I$ with |t - s| < r, there is an $i \in \{1, ..., 2n\}$ are i.i.d., namely X n (k) ~ Berpn , where pn = P[N2-n t \ge 1]. n=1 Similarly, we infer the second inequality in (4.7) from f d $\mu = \infty$ $\mu(\{f \ge n\}) = n=1 \infty$ $\mu(\{f \ge n\}) = n=1 \infty$ 1}). F \in F By Lemma 1.47, u* is an outer measure and u* (A) = $\mu(A)$ for any A \in A. To this end, we apply Kolmogorov's moment criterion (Theorem 21.42 with α) = 4 and β = * 1)., Xn be independent random variables with E[Xi] = 0 and Var[Xi] < ∞ for i = 1, . (2.9) j \in J Proof The class of sets {($-\infty$, b], b \in R} is an \cap -stable generator of the Borel σ algebra B(R) (see Theorem 1.23). For example, we can compute the distribution of a sum of two independent random variables by a simple convolution formula. Hint: First use the inclusion-exclusion formula (Theorem 1.33) to derive a criterion similar to that in Exercise 1.5.4(iii). Let K1 be a finite transition kernel from (Ω0, A0) to (Ω1, A1) and let κ^2 be a finite transition kernel from ($\Omega \times \Omega 1$, $A0 \otimes A1$) to ($\Omega 2$, A2). $s \in I \cap [0,t]$ Consider now the random time $\tau := \sup\{t \in I : Xt \in K\}$ of the last visit of X to K. For general T, the claim follows by linear scaling. Define the family ($PJ : J \subset I$ finite, $0 \in J$) by $PJ := \delta x \otimes k = 0$ Kolmogorov's extension theorem, it is enough to show that this family is consistent. For square integrable random variables X, by the best prediction for X we will understand the F -measurable random variable that minimizes the L2 -distance from X..) = ∞ n=1 3 xn 4-n. 9.1 Processes, Filtrations, Stopping Times 217 Before we present the (simple) formal proof, we state that in particular (i) and (iii) are properties we would expect of stopping times. Define $\beta := \infty$ N=1 N Xi-1 (Bi): B1 $\in \beta$ 1, . Then, at any time, X makes at most one step to the right. Often a useful approximation can be obtained by taking a limit of such distributions, for example, a limit where the number of impacts goes to infinity. \blacklozenge Lemma 1.42 yields uniqueness in Carathéodory's theorem. It remains to show by some standard arguments (LDP 1) and (LDP 2) for arbitrary open and closed sets, respectively. Then the map $\kappa 1 \otimes \kappa 2$: $\Omega \times (A1 \otimes A2) \rightarrow [0, \infty)$, $(\omega 0, A) \rightarrow \kappa 1$ ($\omega 0, d\omega 1$), $d\omega 2$) 1A (($\omega 1, \omega 2$)) $\Omega 1 \Omega 2$ 14.2 Finite Products and Transition Kernels 313 is well-defined and is a σ -finite (but not necessarily a finite) transition kernel from (Ω0, A0) to (Ω1 × Ω2, A1 ⊗ A2). Pp is that probability measure on (Ω2, A2) under which the coordinate maps Yi are independent Bernoulli random variables with success probability p. For example (see [8]), see the following theorem. Clearly, Y is adapted. Proof " = " We construct X as a canonical process. By the uniqueness. theorem (Lemma 1.42), we thus have equality for all $B \in B(R)$. In other words, the Dirichlet problem (19.4) has a unique solution given by (19.1)). (i) Which p maximises the entropy? Proof (i) The statement is obvious if f = 1A is an indicator function. are independent and Ξ -distributed. Hence the distribution of X is characterized by its moments. Metropolis Algorithm We have seen already in Example 17.19 how to simulate a Markov chain on a computer. By T we denote the set of trifurcation points, and let TL := T ∩ BL. (ii) Weak convergence (as introduced in Definition 13.12) induces on Mf (E) the weak topology τw . Give an example that shows that the monotonicity of ϕ is essential. In order for the series to converge almost surely, it is sufficient (and also necessary, as a simple application of Kolmogorov's three-series theorem, $\phi r = er\psi = \lim n \rightarrow \infty ern(\phi n - 1)$ is a CFP. 4 19.3 Finite Electrical

Networks An electrical network (E, C) consists of a set E of sites (the electrical contacts) and wires between pairs of sites. As the individual experiments ought to be independent, we should have for any choice $\omega 1$, Then E[X] = $\theta \propto 0 x e - \theta x dx = \theta - 2$. (7.5) Hence $g \in L1$ (ν) if and only if $gf \in L1$ (µ), and in this case (7.5) holds. 191 195 203 9 Martingales ... Theorem 6.19 For finite µ, $F \subset L1$ (µ) is uniformly integrable if and only if there is a measurable function H : $[0, \infty) \rightarrow [0, \infty)$ with limx $\rightarrow \infty$ H (x)/x = ∞ and sup $f \in F$ H (|f|) dµ < ∞ . [149, Theorem 8.5]. be identically distributed could be replaced by the condition that the variances be bounded (see Exercise 5.3.1). Define the local energy level of a single atom at $i \in \Lambda$ as a function H i of the state x of the whole system, 1 2 H i (x) = 1 - P[Z > x] *) = 1 - P[Z > x] *) = 1 - P[Z > x] *) = 1 - P[Z > x] * (x) = 1 + P[Z > x] *) i=1 We interpret Zn as the size of a population at time n and Xn, i as the number of offspring of the ith individual of the nth generation. For any $A \in A$ and $n \in N$, we thus get $\mu(A \cap En - 1) \cap \Omega i = \nu(A \cap Ei - 1) \cap \Omega i = \nu(A \cap En - 1) \cap \Omega i = \nu(A \cap Ei - 1) \cap \Omega i = \nu(A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En - 1) \cap \Omega i = (A \cap En$ measures μ and ν are called equivalent (symbolically $\mu \approx \nu$) if $\nu 0 \mu$ and $\mu 0 \nu$. 0, if θ (p) = 0, Theorem 2.43 For any $p \in [0, 1]$, we have $\psi(p) = 1$, if θ (p) > 0. (ix) (Binomial distribution) By the binomial theorem, $\phi(t) = n - n (1 - p)n - k$ (peit)k = (1 - p + peit)n. The main goal of this section is to express the probability $1 - F(x_1, x_1)$ that the random walk never returns to x1 in terms of effective resistances in the network. These balls, as well as their closures, are subsets of A1 .) Let UD1 := { $U \in U : U \subset D1$ }. Intuitively, (17.15) suggests that we define pt = et q in a suitable sense. Xsn = in] > 0, we have *)*) P Xt = i Xs1 = i1, Then aX ~ exp θ/a . Example 1.58 Important special cases for the Lebesgue measure on R. (9.5) This is the discrete analogue of the celebrated Black-Scholes formula for option pricing in certain time-continuous markets. (i) For any $x \in E$, the map $\omega \to f(\omega, x)$ is in L1 (μ). Finally, ϕ is called Hölder-continuous of order γ if there exists a C such that (21.1) holds for all s, $r \in E$. Hence, if Fn $\to F$, then F (∞) = limn $\to \infty$ Fn (∞). , Ad-1 are the disjoint coset classes of the normal subgroup)r* Ω . The first two derivatives of F are F (λ) = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ) = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and F (λ = $-E[Xe - \lambda X]$ and $E[(X2)e-\lambda X]$. t in t \in Lp (P) such that Xtn n $\rightarrow \infty - \rightarrow X$ Assume that, for every t ≥ 0 , there exists an X p L. Proof This is trivial! Theorem 8.6 (Summation formula) * Let I be a countable set and let (Bi)i \in I be) pairwise disjoint sets with P i \in I Bi = 1. + T s 1 Ln +1 > T1 +. Whether or not { $\tau \leq t$ } is true can thus be determined on the basis of the information available at time t. By the law of large numbers, we have Tt1 \approx t/D for large t. If it is chosen such that as a function of B it is a probabilities. (iii) Let A, B \in A be such that $\mu(A = B) = 0$. We call this coupling the independent coalescent. 3. We do not strive for the greatest generality but rather content ourselves with the key theorem 2.37) implies that $\psi(p) = P[A] \in \{0, 1\}$. Theorem 2.37) implies that $\psi(p) = P[A] \in \{0, 1\}$. Theorem 2.37) implies that $\psi(p) = P[A] \in \{0, 1\}$. is ergodic if and only if any I-measurable f: $(\Omega, I) \rightarrow (R, B(R))$ is P-almost surely constant. In particular, f is strictly convex and hence assumes its p (unique) minimum at x0 = y 1/(p-1). Using Markov's inequality (Theorem 5.11), we estimate) *) * P[Sn \ge 0] = P e \tau Sn \ge 1 \le E e \tau Sn = \phi(\tau) = n \cdot 3 Let $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ as in (ii). Then there exists an $A\epsilon \in A$ with $0 < \mu(A\epsilon) < \infty$ such that $A\epsilon \subset |f| > (1 - \epsilon)f \infty$. Furthermore, and more importantly, we use Rayleigh's monotonicity principle to show that if a random on a graph is recurrent, then it is also recurrent on any subgraph. That is, $Px [\lim_{n \to \infty} Xn = 1] = x = 1 - Px [\lim_{n \to \infty} Xn = 0]$. We
first show 3the statement for nonnegative f. Furthermore, the uniform distribution UE is the unique invariant distribution. (ii) Conclude that for any random variable Y on [0, 1], the distribution is uniquely determined by its moments mn := E[Y n], n \in N., XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1, . Corollary 12.18 Let X = (Xn) n \in N. , XT is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random variables D1 is called binary splitting or a binary model if there exist random varia Hence f is B(I)* -measurable. (ii) Show that (Ω , A, $\delta\omega0$) is complete. It makes the following assumptions: • Atoms are placed at the sites of a lattice Λ (for example, $\Lambda = \{0, ... \in \Omega, define the set of countable coverings F with sets F \in \Lambda$: $U(A) = F \subset A : F$ is at most countable and $A \subset F \in F$ Define * $\mu(A) := \inf F \in U(A)$, $F \in U(A) = F \cap F \cap F \cap F \cap F$ and every open B and every open A $\subset E, \mu * (B)$ $\cap A$) + $\mu *$ (B c $\cap A$) $\leq \beta(A)$. 23.1. Here we have developed the abstract framework (principle of large deviations) for the description of the speed of concentration for a sequence of probability measures. Here our first goal is to change \cdot p into a proper norm for all $p \in [1, \infty]$. Due to the continuity of φ for all $\epsilon > 0$, we thus have inf $|\varphi(t) - 1| : t \in [-\pi, \infty]$. π)D \ ($-\varepsilon$, ε)D > 0. Then 1 n $\rightarrow \infty$ Xk $\rightarrow E[X0 | I]$ in Lp (P)., Xin)] = L[(Xj1, .(1.9) 20 1 Basic Measure Theory Hence (Ω , d) is a compact metric space. Then Y is a random walk with transition matrix pN. The strategy is to define a number $\mu * (E)$ for each $E \in 2\Omega$ by covering E with elements of E and then determine the total content. j = 1 This lemma allows us to make the following definition (since the value of I (f) does not depend on the choice of the normal representation). In order to check the assumptions of Theorem 1.53, we have to show that μ is σ -subadditive. Definition 9.6 If X is a random variable (or a stochastic process), we write L[X] = PX for the distribution of X. The σ algebra B(Ω) $:= B(\Omega, \tau) := \sigma(\tau)$ that is generated by the open sets is called the Borel σ -algebra on Ω . Hence, by Theorem 2.13(iii), it is enough to show that (Zk)k K is independent. Let Gy, x 1, x 2, x 3 be the event where in EL exactly those edges are open that belong to these three paths (that is, all other edges in EL are closed). , L - 1 d for some L $\in \mathbb{N}$. , CN \in C. On the other hand, for all $n \in N$, the difference Sn - Tn is deterministic, contradicting the assumption that (iii) does not hold. In this section, we introduce names for classes of subsets of Ω that are stable under certain set operations and we establish simple relations between such classes. Reflection Why do we assume σ -finiteness of the measures in the Radon-Nikodym theorem?, tN) is a strong Markov process (by Theorem 17.14); hence we have) *) * Ex F (BT n +t) t ≥ 0 FT n = Ex f (BT n +t) t = 0 FT n = Ex f (BT n +t) t ≥ 0 FT n = Ex f (BT n +t) t ≥ 0 FT n = Ex f (BT n +t) t = 0 FT n = Ex f (BT n +t) t ≥ 0 FT n = Ex f (BT n +t) t = 0 with $VT = v0 + (H \cdot X)T$., Xn be independent random variables with Xi = Berpi for any i = 1, (i) Let $\mu, \mu 1, \mu 2$, Therefore $x \rightarrow \psi \omega$ (x) is also right continuous. Theorem 1.33 (Inclusion-exclusion formula) Let A be a ring and let μ be a content on A. n Yn = i=1 We show that $(Y-n)n\in N$ is an F-backwards martingale, such that Fnl (gl) $k\in N$ converges k for all $l \in N$. $n \rightarrow \infty$ Proof We have shown already the convergence of the finite-dimensional distributions. In fact, if $P[|Xn,l| > \epsilon] < \delta$, then we have $|\phi n, l(t) - 1| \le 2\epsilon + \delta$ for all $t \in [-1/\epsilon, 1/\epsilon]$. As in Theorem 1.23, it can be shown that $\sigma(A) = B(\Omega, d)$. M This formula can be derived formally via a small computation with conditional probabilities. For s, t \in I, by assumption, $|t - s| n \le \varepsilon$ and thus n fs + $(t - s) k - 1 |f(t) - f(s)| \le n n k = 1 \le C(\varepsilon) n 1 - \gamma |t - s|\gamma = C |t - s|\gamma$. Then E is a σ -compact metric space and therefore in particular, separable. (i) Show that dH is a metric that induces convergence in measure. By the strong law of large numbers, there is an n - 0 = n0 (ω) with $1 \text{ k}-1 = \exp - k = -n 1$ log(k) for all $n \ge n - 0$. By Exercise 8.2.2, we have $E[X | F] \in I \text{ a.s.}$, hence $\phi(E[X | F])$ is well-defined. For every $\varepsilon > 0$, there exists an $N \in N$ and a $A^{\tilde{c}} \in E \{0, \dots, N\}$ such that, letting $A\varepsilon = A^{\tilde{c}} \varepsilon \times E \{N+1, N+2, \dots\}$, we have $P[A | A\varepsilon] < \varepsilon$. \diamond Example 20.26 Let I = N0 or I = Z, and let (Xn) $n \in I$ be an i.i.d. sequence with values in the measurable space (E, E). Then p is a stochastic matrix and $q = \lambda(p - I)$. Takeaways A branching process dies out eventually if the mean number of offspring is no larger than 1. We use this to infer uniqueness of the electrical current. Determine the average code length of a letter and compare it with the entropy H3 in order to check the efficiency of the Morse code. Later we will see that the following theorem is valid in greater generality. Remark 7.31 Clearly, $\mu \perp \nu$ Example 8.31 Let X and Y be real random variables with joint density f (with respect to Lebesgue measure $\lambda 2$ on R2). In particular, it has to be shown that limt $\uparrow 1$ Yt = 0 almost surely. 6 and 7. The corresponding Markov chain X^{-} is transient, and Δ is the only absorbing state. Indeed, let N > 0 be such that $\lambda = (Xt) t \ge 0$ be an F-supermartingale such that $t \rightarrow of X$ with RCLL paths. Further, show that strict inequality can hold in the lower bound (LDP 1)., n - 1, let n - 1 fi (ωi) = κjk , jk+1 (ωi , $Aji+1 \times \cdots \times Ajn$). Lemma 13.10 Let (E, d) be a metric space. If $\lambda(\{f = 0\}) > 0$, then (since $\mu(\{f = 0\}) > 0$, then (since (\mu(\{f = 0\}) (see $n \rightarrow \infty$ 2.1 Independence of Events 59 Remark 1.14). A stochastic process $X = (Xt) t \in I$ is called a time-homogeneous Markov process on the probability space (Ω , A, Px) with Px [X0 = x] = 1. (ii) For any subsequence of (fn) $n \in N$, there
exists a subsubsequence that converges to f almost everywhere. Which of the functions $f(x) = 1\{0\}$ (x), g(x) = 1R(Z(x), h(x) = sin(1/x) for x = 0 and h(0) = -1, are lower semicontinuous? "(i) \Rightarrow (ii)" By Prohorov's theorem, (Pn)n $\in \mathbb{N}$ is relatively sequentially compact. That is, a sequence (μ n) $n \in \mathbb{N}$ is relatively sequentially compact. That is, a sequence (μ n) $n \in \mathbb{N}$ is relatively sequentially compact. That is, a sequence (μ n) $n \in \mathbb{N}$ is relatively sequentially compact. That is, a sequence (μ n) $n \in \mathbb{N}$ is relatively sequentially compact. $d\mu n \rightarrow f d\mu$ for any bounded continuous function $f: E \rightarrow R$. By the Markov property, for every $n \ge N$, '() * $P\pi A\epsilon \cap \tau - n$ (B) = $P\pi (X0)$, Note, however, that this does not exclude the $n \rightarrow \infty$ with positive probability; for instance, if Sn grows like \sqrt{n} . Then $X\tau = a$ if $\tau \le n$. \diamond 202 8 Conditional Expectations Takeaways The conditional expectation of a random variable X given a σ algebra F is the best prediction on X that can be made given the information coded in F (at least if X has a second moment). 19.15. However, what is the probability that A occurs if we already know that B occurs? However, often there is a version with right continuous paths that have left-sided limits. Then (Yn)n \in N variables with $E[|X0|p] < \infty$. E[Xt] is right continuous. $f \in F$ (ii) There is a function $0 \le h \in L1$ (μ) such that $h d\mu < \delta(\epsilon)$. Indeed, for the closed set ($-\infty$, 0], we have limn $\rightarrow \infty \delta 1/n$ (($-\infty$, 0]) = $0 < 1 = \delta 0$ (($-\infty$, 0]). ., the following strengthening holds, lim sup $2 n \rightarrow \infty$ |Sn | 2n Var[X1] log(log(n)) = 1 almost surely. By Example 18.7, we can construct independent successful couplings (X(i), Y (i)), i = 1, . For further reading, see, for example [103] or [88]. Remark 21.10 The covariance function determines the finite-dimensional distributions of a centered Gaussian process since a multidimensional normal distribution is determined by the vector of expectations and by the covariance matrix. 21.4 Supplement: Feller Processes . 250 11 Martingale Convergence Theorems and Their Applications Fig. Takeaways Consider an exchangeable family X1 , X2 , . (9.3) By construction, XT + - XT - = 0 if VT + - VT - = 0. & Exercise 5.1.5 Let X1 , X2 , . Then: (i) σ V τ and $\sigma \wedge \tau$ are stopping times. $\in \mathbb{R}$ such that $\mu = n = 1 \delta xn$ is a σ -finite measure. (ii) Let $X \sim exp\theta$ and a > 0. Clearly, $Q \mid P \mid f \mid d\nu \mid \nu \mid 0 \mid \mu$. measure (or, briefly, in measure), symbolically fn $\rightarrow f$, if $n \rightarrow \infty \mu(\{d(f, fn) \mid n \in N \mid n$ $> \epsilon \} \cap A$) $- \rightarrow 0$ for all $\epsilon > 0$ and all $A \in A$ with $\mu(A) < \infty$, and a.e. (ii) μ -almost everywhere (a.e.), symbolically fn $- \rightarrow f$, if there exists a μ -null set $N \in A$ such that $n \rightarrow \infty d(f(\omega), fn(\omega)) \rightarrow 0$ for any $\omega \in \Omega \setminus N$. 381 17 Markov Chains. Thus, a time-homogeneous Markov process is simply a stochastic process with the Markov property and for which the transition probabilities are time-homogeneous. We are now interested in properties of this process X that cannot be described in terms of finite-dimensional distributions but reflect the whole path t \rightarrow Xt. (5.15) n=1 n -1 Then lim sup an Xk = 0 almost surely. , ωn . Then we come to finite products of measure spaces and product measures with transition kernels. A stochastic kernel κY , F from (Ω , F) to (E, E) is called a regular conditional distribution of Y given F if κY , F (ω , B) = P[{Y $\in B$ }|F](ω) for P-almost all $\omega \in \Omega$ and for all $A \in F$, $B \in E$., Wd are independent and N0,1 -distributed. Definition 4.16 For measurable f: $\Omega \rightarrow R$, define $f p := 1/p | f | d\mu p$, if $p \in [1, \infty)$, and $f \infty := \inf K \ge 0$. (4|f| > K) = 0. $(14.9) \Omega 2$ Proof For $f = 1A1 \times A2$ with $A1 \in A1$ and $A2 \in A2$, the statement is true by definition. The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland Preface to the Third Edition New in the third edition: the sections close with a short "takeaways" block where highlights of the section are summarized sometimes on an informal level without full rigor., km) \in Nm 0 with k1 + . In order to do, criteria for relative compactness of probability measures on C([0, ∞)) are needed. For functions of exchangeable random variables X1 , X2 , . As shown above, we have P[L = ∞] = 0. Then $\mu(E)$ = sup fN d μ $N \in N = \sup \lim N \in N = \sup \lim N \in N$ and let $f: \Omega \to [0, \infty)$ be a measurable map. As C is compact, there exists an $n \in N$ with $C \subset i=1$ Ai . 2.1 Percolation on a 15×15 grid, p = 0.42. (Here Pnk := Pn $\circ \pi k - 1$, where $\pi k : Rd \to R$ is the projection on the kth coordinate.) Let ek be the kth unit vector in Rd. Theorem 19.30 Let C and C be edge weights on E with C $(x, y) \le C(x, y)$ for all x, $y \in E$. A map $f: \Omega \to R$ is and let $\mu = \omega \in \Omega$ $|f(\omega)| \alpha \omega < \infty$. Then $X^* := \lim \text{ supn} \to \infty$ Xn are almost surely constant. Let Z1, Z2, Let $B \subset A$ be measurable with $\lambda(B) > 0$. Further, let (pe)e $\in E$ be a probability vector. By permute the E1; hence $y \in E1 \cap E1$., D and let Yt = (Yt1, D) and let Yt =running the query. An elegant way to decouple the coordinates is to pass from discrete time to continuous time in such a way that the individual coordinates become independent but such that the Green function remains unchanged. $\epsilon \rightarrow 0 \ \epsilon \rightarrow 0$ We say that the individual coordinates become independent but such that the individual coordinates become independent but function I if (LDP 1) and (LDP 2) hold with $\varepsilon_n = 1/rn$ and $\mu/rn = Pn$. Hence the error is at most of order n-1/2. 480 19 Markov Chains and Electrical network on Zd with unit resistors between neighboring points., Zn }. Let p 1 C (x) \cap C p (x 2) = $\emptyset \cap$ #C p (x 2) = \emptyset \cap BL-1 2.4 Example: Percolation 81 be the event where there exist two points on the boundary of BL that lie in different infinite open clusters. lim sup Fnk (x) \leq F (x). Example 18.18 (Ising model) In the Ising model described above, we have x-i = {x i, -1, x i, +1}... Theorem 20.14 (Individual ergodic theorem, Birkhoff [16]) Let f = X0 \in L1 (P). We prepare for the proof of the CLT with a lemma. I would like to take the opportunity to thank all of those who helped in improving the first edition of this book, in particular: Michael Diether, Martin Kolb Manuel Mergens, Thal Nowik, Felix Schneider, Wolfgang Schwarz, and Stephan Tolksdorf. As an excuse for presenting this section in a chapter on Markov chain in order to prove a theorem on the stochastic order of binomial distributions. At that point it was only a small step to show that the Lebesgue measure is regular in the sense that the measure of an arbitrary measurable set can be approximated by compact subsets as well as by open supersets. t $\rightarrow 0$ By Theorem 15.32, this implies E[X2n] = (-1)n $\phi(2n)(0)$. Hence $\nu^{\tilde{}} := h\nu$ is a finite measure and $\nu = h-1 \nu^{\tilde{}}$ is uniquely defined by ν . (ix) The binomial distribution bn,p with parameters $n \in N$ and $p \in (0, 1)$ is not infinitely divisible (why?). We interpret the edges as tubes along which water can flow. From this an outer measure will be derived. Define $R^- := \{X \text{ and } Y \text{ are right continuous}\}$ and choose an $R \in A$ with $R \subset R^-$ and P[R] = 1., $n\}$ with conductances C(k - 1, k) > 0 and C(k, l) = 0 if |k - l| > 1. This map can be one of the edges as tubes along which water can flow. to one, as with linear maps and matrices, or it may map only some properties uniquely, as with matrices and determinants. (i) (ii) Since $\emptyset \in A$, we have $\{\emptyset\} \in U(\emptyset)$; hence $\mu * (\emptyset) = 0$. A metric d on E is called complete if any Cauchy sequence with respect to d converges in E. Example 7.38 In the converse implication of the theorem, the assumption of finiteness is essential. More precisely, for any $\varepsilon > 0$, we have +, 1 V P Sn $\ge \varepsilon \le 2$ for all $n \in N$. $0 \le 1 - \operatorname{Re}(\varphi X(t)) \le 4(1 - \operatorname{Re}(\varphi X(t))) \le 4(1 - \operatorname{Re}(\varphi$ limes superior (see Definition 1.13). Furthermore, let $F: R \rightarrow R$ be monotone increasing and right continuous. For $n \in N$, $t \in N0$ and l = 1, +Xn and m := E[Sn]. Proof Apply Theorem 14.25 with $\kappa 2 = \kappa$ and $\kappa 1$ ($\omega 0$, \cdot) = μ . If (E, d) is a metric space and A, $B \subset E$, then we write $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ and $d(x, B) := d(\{x\}, B)$ for $x \in E$. = μi $(-\infty, x] \cdot j = i$ Hence indeed PXi = μi . If x is a state and $i \in \Lambda$, then define $x-i := \{y \in E : y(j) = x(j) \text{ for } j = i\}$. (i) C := sup $|f| d\mu < \infty$. Show that Uf $\in B(\Omega 1)$. Show that Uf $\in B(\Omega 1)$. Exercise 13.2.10 Let X, X1, X2, . Then (Xn n=1,...,N is exchangeable. Thus ('' ($\otimes k = \lim E E F d\Xi \infty f1 d\xi n (X) \cdots fk d\xi n (X) \rightarrow \infty *$) = E f1 (X1) $\cdots fk d\xi n (X) \rightarrow \infty *$) = E f1 (X1) $\cdots fk d\xi n (X)
\rightarrow \infty *$) = E f1 (X1) $\cdots fk d\xi n (X) \rightarrow \infty *$) and let (fn)n \in N be a sequence of nonnegative functions in L2 (Ω , A, $\mu + \nu$) with fn \uparrow f. \geq 0 such that gn := n i=1 n \rightarrow \infty \alpha i 1Ai $-\rightarrow$ f. \clubsuit 302 13 Convergence of Measures Since E is Polish, PX1 is tight. The unique solution of this differential equation is f1 (t) = 1 - e - t. The existence of the convolution semigroup follows by Corollaries 16.8 and 16.7 if we define μr by ϕr . Without proof, we present the following topological result (see, e.g., [37, Theorem 13.1.1]). \blacklozenge 200 8 Conditional Expectations Example 8.19 Let X1, . (21.19) $n = -\infty$ If f is the density of a probability distribution on R with characteristic function ϕ and supx \in R x 2 f (x) < ∞ , then the Poisson summation formula holds, $\infty \infty f(s + n) = -\infty$ $n = -\infty \phi(k) e^{2\pi i s}$ for all $s \in \mathbb{R}$. Xin) has the n-dimensional normal distribution with $\mu = \mu I := (\mu i 1, \ldots 1.2 \text{ Set Functions } 15 \infty$ (iv) Let A be a ring and let $A = An \in A$. Further, for any $p \in [1, \infty]$, define the vector space Lp (μ) := f : $\Omega \to \mathbb{R}$ is measurable and f $p < \infty$. Then, find the search box and enter the name of the person or business you want to call.Results to ExpectIf you're searching in the Telkom directory, expect to find the name, address and phone number of the party you want to call, if they have a listed number. Definition 7.1 (Factor space) For any $p \in [1, \infty]$, define Lp $(\Omega, A, \mu) := Lp (\Omega, A, \mu)/N = \{f := f + N : f \in Lp (\mu)\}$. Manifestly, we can choose Htn = Ht 1{|Ht | (ii) Compute fn d λ and determine f d λ as a limit of integrals. A simple application of Jensen's inequality yields H (μ) ≥ 0 (see Lemma 5.26 and Exercise 5.3.3). By Example 17.60, it is enough to consider the smallest p2 that fulfills (17.31). For X \in L1 (P), we disjoint events with i \in I define a map E[X |F]: $\Omega \rightarrow \mathbb{R}$ by E[X |F] (ω) = E[X |Bi] \Rightarrow Bi ω . We will $n \rightarrow \infty$ come back later to the point that this superficially contrasts with Sn $\rightarrow 1$ a.s. (see Example 11.6). Now let E = {0, 1} and let X1, X2, . (iii) A is closed under countable unions. Hence we are in the situation of drawing colored balls without replacement. Cramér's theorem says that limn $\rightarrow \infty$ n1 log(Pn ([x, ∞))) = -I (x) for x > 0 and (by symmetry) $\lim n \to \infty x > 0$, 1 n log(Pn (($-\infty, x$])) = -I(x) for x < 0. = Dn -1 = -1 and Dn = 1 = p(1 - p)n - 1. Then M -M = A - A is a predictable martingale; hence (see Exercise 9.2.2) Mn -Mn = M0 - M0 = 0 for all $n \in N0$. We say that X has almost surely continuous, if for almost all $\omega \in \Omega$, the path $t \to Xt(\omega)$ is continuous. Case 2: limz 1 ψ (z) > 1. For λ -systems this is not true in general. "(vii) \Rightarrow (v)" Let G be open and ε > 0. In particular, this is true if A is a ring. In order to compute u(x), we replace the network step by simpler networks such that the effective resistances between 0, 1, and x remain unchanged. n=1 Example 1.30 (Contents, measures) (i) Let $\omega \in \Omega$ and $\delta \omega$ (A) = 1A (ω) (see (1.2)). Show that $f \in L1$ (μ) and fn - f d μ = lim fn d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d μ - f d μ = lim fn d\mu = lim fn d μ = lim fn d μ = lim fn d μ = l holomorphic in $\{z \in C : \text{Re}(z) \in (1, s)\}$ (and is thus uniquely determined by the values $\varphi(r)$, $r \in (1, 1 + \varepsilon)$ for any $\varepsilon > 0$). Letting $\varepsilon \downarrow 0$, we get a Borel σ -algebra, that can also be generated using simple sets such as rectangles. (5.9) By the monotone convergence theorem (Theorem 4.20), we have) * $n \rightarrow \infty$ E[Yn] = E X1 1{X1 ≤ n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 < n} $\rightarrow \infty$ E[Yn] = E X1 1{X1 8.20 (Jensen's inequality) Let $I \subset R$ be an interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be
convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\Phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\Phi: I \to R$ be convex and let X be an I-valued random variable on (Ω , A, P). $h \to 0$ (interval, let $\Phi: I \to R$ be convex and let X be an I-valued random variable on (I \to R) (interval, I). $h \to 0$ (interval, I) (I \to R) (interval, I) (I \to R) (I on entropy of p. A market in which every claim can be replicated is called complete. Clearly, it is necessary that π be the unique invariant distribution; that is, up to a factor π it is the unique left eigenvector of p for the eigenvalue 1. Further, let $\Omega 1 = [0, 1]$, let A 1 = B([0, 1]) be the unique invariant distribution; that is, up to a factor π it is the unique invariant distribution on [0, 1]. On the other hand, if a family of stochastic kernels Ks,t, s < t, fulfills the a minimal consistency condition (the Chapman-Kolmogorov's extension theorem allows to construct a probability space and a Markov process on it that fits to these kernels. (For finite A, the claim is trivially true even for p = ∞ .) For example, let $\Omega = N$, $A = 2\Omega$ and let μ be the counting measure. Now let a = 1, $b = \theta$ and c = -E[X] - b E[Y]. We have to show that there exists an $N \in A$ with $\mu(Uf) = 0$. Here Helly's theorem is the tool. 1.5 Random Variables 47 (ii) Let $p \in [0, 1]$ and $n \in N$, and let $X : \Omega \to \{0, ..., 0\}$. Here we have studied the properties of this space as a topological space and as a measure space. If this is the case, then $\alpha \in \mathbb{R}$ and for $A \subset \Omega + A \in \mathbb{R}$, we would have $\alpha \ge \phi(\Omega + A) = \phi(\Omega + A) = \phi(\Omega + A) = \phi(\Omega + A)$. The time-homogeneous Markov property is immediate from the fact that the increments are independent and stationary. ♦ Remark 6.5 Almost everywhere convergence and convergence in measure determeas mine the limit up to equality almost everywhere. 12.1. Hence we shall show that a countably infinite exchangeable family of random variables is an i.i.d. family given the exchangeable σ -algebra E. (ii) If τ is measure-preserving and I is P-trivial, then (Ω, A, P, τ) is called ergodic. If (Yn)n=0,1,...,T is an F-martingale, then Yn = E[YT Fn] for all $n \leq T$. If we let g = 1 $\mu(A\epsilon)$ 1A ϵ , then g1 = 1 and $\kappa(f)1 \geq fg1 \geq (1 - \epsilon)f \infty$. 2 Let E = Z (with the discrete topology) and let Xt = t Yn for all $t \in N0$. For the so-called Gibbs sampler or heat bath 18.3 Markov Chair Monte Carlo Method 451 algorithm, the idea is to adapt the state locally to the stationary distribution. k Then PX =: bn,p is called the binomial distribution with parameters n and p; formally bn,p n n k = p $(1 - p)n-k \delta k$. A stochastic process X = $(Xn)n \in -N0$ is an F-martingale with respect to F if X = $(X - n)n \in -N0$ is an F-martingale. Since Z is a tree (that is, it is connected and contains no circles), we have #Z - 1 = 1 degHL (h). Accordingly, fix (x1, y1), (x2, y2) $\in E \times E$. (i) Conclude the statement for X with a continuous density. , 6}2, A2 = {1, . By Theorem 8.37 (with $E = \{0, 1\}_n \subset Rn$), a regular conditional distribution exists: $\kappa Y_X(x, \cdot) = P[Y \in \cdot |X = x]$ for $x \in [0, 1]_n$. Subtracting the differences for each time step, we decompose a submartingale into a sum of a martingale and a monotone increasing predictable process. A metrizable space (E, τ) is called separable if there exists a countable dense subset of E. Hence (by Theorem 7.11) p p (' E[X]1/p + E[Y]1/p $- \rightarrow 0$. Hint: Show that Θ and Φ are independent, and compute the distributions of Θ and Φ . s It is easy to check that $\lim s \downarrow 0$ 1 pt +s (x, y) - pt (x, y) = (q \cdot pt)(x, y)., kr $\in N(x, x)$ with gcd($\{k1, ..., kr \in N(x, x) \mid r \in N(x, y) = (q \cdot pt)(x, y)$. mutually disjoint subsets of I . 25.5 Recurrence and Transience of Brownian Motion.. For such f, define the matrix M = . Now $(f + g)p \le 2p$ (f p + g p); hence $f + g \in Lp(\mu)$. As Cc (E) is a separating class for $M \le 1$ (E) (see Theorem 13.11), (i) follows by Theorem 13.11), (i) follows by Theorem 13.34. (Here $xk = x(x-1)\cdots(x-k+1)$ for $x \in R$ and $k \in N$ is the generalized $f r \in N$, then one can show as in the preceding example - is the distribution of the waiting time for the rth success in a series that br,p of random experiments. (viii) Let X and Y be independent with X ~ N0, \sigma 2 and Y ~ $\Gamma\theta$, r, where $\sigma 2$, θ , r > $\sqrt{0}$. By $n \rightarrow \infty$ Lemma 15.47, this is equivalent to lim = eit x - 1 and (Xn, I) is centered. Denote the results by Y1, Using the contraction principle this large deviations principle can be reduced to functions of the random variables. Corollary 1.83 (Trace of a generated σ -algebra) Let $E \subset 2\Omega$ and assume that $A \subset \Omega$ is nonempty. Show that then there would be a sequence (Cn)n \in N with Cn $\uparrow \infty$ and 0 = 1w-lim(Cn ϕ n). The empirical distributions converge to the distribution of the random variables. They are of different levels of difficulty indicated by the number of clubsuits. We have to show that ϕ attains the maximum α ; that is, there exists an $\Omega + \epsilon$ A with $\phi(\Omega +) = \alpha$. We start with a preliminary lemma. n (23.11) We use the method of an exponential size-biasing of the distribution $\mu := PX1$ of X1, which turns the atypical values that are of interest here into typical values. A function $f: E \to R$ is called harmonic on $E \setminus A$ if pf(x) = f(x) for all $x \in E \setminus A$. Here we follow the proof of Burton and Keane [23] as described in [63, Section 8.2]. Takeaways A Riemann integrable function on a compact interval is Lebesgue integrable and the integrals coincide. .), by the approximation theorem for measures, there exists a sequence of measurable sets (Ak) $k \in N$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, . * Exercise 7.2.3 Let X be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, ..., A be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, ..., A be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, ..., A be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, ..., A be a real random variable and let p, $q \in (1, \infty)$ with $Ak \in \sigma$ (X1, ..., A be a real random variable and let p) a real random variable and let p (X1, ..., A be a real random variable and let p) a real random variable and let p (X1, ..., A be a real random variable and let p) a real random variable and let p (X1, ..., A be a real random variable and let p) a real . Then A0 A1 (w1 - w0) I (A1) = 1 (w(x) - w(y)) I (x, y). If we remove the superconductors from the network, we end up ($0 \leftrightarrow \infty$) is not smaller than that of with the network of Fig. Let $p \in (1, \infty)$. Substituting z = (1 - it)x, we get $1 \varphi(t) = \Gamma(r) \propto x r - 1 - x$ it $x \in 0$ (1 - it) -r dx = $\Gamma(r) 3$ Hence, it suffices to show that $\gamma 0, \infty, t$ zr - 1 = -z dz. Perfect Sampling The MCMC method as described above is based on hope: We let the chain run for a long time and hope that its distribution is close to the invariant distribution distribut Show that the states 6, 7 and 8 are positive recurrent and compute the expected first entrance times E6 [$\tau 6$] = 17, 4 E7 [$\tau 7$] = 17 5 and E8 [$\tau 8$] = 17, (7.3) On the other hand, for any measurable f : $\Omega \rightarrow [0, \infty)$, equation (7.3) defines a measure ν on (Ω , A). If n is
sufficiently large that mn < 12, then k kn n it Xn, $\log(\phi_{n,1}(t)) - \phi_{n,1}(t) - 1$ $E[e-1] = \log \phi n(t) - l = 1 = 1 \le kn \quad \phi n, l(t) - 1 = 1 \le n \quad kn \quad \phi n, l(t) - 1 = 1 \le n \quad kx, F(\omega, Ai) P[d\omega] \quad \alpha i = 1 = n \quad \alpha i = 1 = n \quad B i = 1 = n \quad A i = 1 = n \quad B i = 1 = n \quad A i = 1 = n \quad A$ $(x) \kappa X, F(\omega, dx) P[d\omega]$. These are some of the most important distributions in probability theory, and we will come back to these examples in many places., kr is called the Frobenius problem. It is sufficient to consider $l \ge 1$ since we get the l = 0 term from the fact that the probability measure has total mass one. For this we would have to rely on prophecy. Assume N takes values in N0 and has the probability generating function $fN \cdot i=1$ Letting $n \to \infty$ and using the σ -subadditivity of $\mu *$, we conclude $\mu * (E \cap Ac) + \mu * (E \cap Ac) + \mu * (E \cap Ac) + \mu * (E \cap Ac)$. We will show (i) F is the distribution function of a (sub-) probability measure. Hence the edges of u lie in an infinite open cluster of K p and there is at least one edge $k \in u$ incident to a point at the boundary BL \ BL-1 of BL. Every A \in A is closed and thus compact. Evidently, Xt - Xs = t - s X1 ~ N0,t - s for all t > s ≥ 0 . A is closed and thus compact. Evidently, Xt - Xs = t - s X1 ~ N0,t - s for all t > s ≥ 0 . M - sk, N - M + sk - k, if x = 1, if x = 0. Then $\phi n = |\phi|n$ is n-fold divisible; however, the factors are not unique. Use the preceding theorem to show the conditional version of Hölder's inequality: *1/p) q *1/q *) E |Y| F E |XY| F ≤ E |X|p F almost surely. For fixed $\omega 0$, by the monotone convergence theorem, the map $A \rightarrow \kappa 1 \otimes \kappa 2$ ($\omega 0$, A) is σ additive and thus a measure. Apply Hölder's inequality to $f \cdot (f + g)p - 1$ and to $g \cdot (f + g)p - 1$ to get p + gp = (f + g)p + 1 du = f(f + g)p - 1 differentiable with derivative $F(x) = f(\omega, x) \mu(d\omega)$. Letting $f := \sup\{fn : n \in N\}$, the monotone convergence theorem yields $f d\mu = \sup fn d\mu \ge \lim fn d\mu = \lim$ $\mu(Bi)$. 113 113 121 125 135 139 xi xii Contents 6 Convergence Theorems . Let A \subset E. 5.4 Speed of Convergence in the Strong LLN . This statement will now be generalised to an arbitrary finite number of sets. We saw that, in general, this is not possible if the differences Xn+1 – Xn take three (or more) different values. Assume that in K (enumerated) urns there are a total of N indistinguishable balls. For any $n \in N$ choose an open set An \supset Gn with $\beta(An) < \mu * (Gn) + \epsilon/2n$. Since ψ is continuous, we infer $\psi(q) = \psi$ lim $qn n \rightarrow \infty = \lim \psi(qn) + \epsilon/2n$. between these points R (0, 1), R (0, x) and R (1, x). We have seen a formula for the first and second moment of a sum of random variables, even if the number of summands is random variables, even if the number of summands is random variables. \in No this end, we compute some of its moments and then use the Kolmogorov-Chentsov theorem (Theorem 21.6). Let U \subset E be closed. \in No this end, we compute some of its moments and then use the Kolmogorov-Chentsov theorem (Theorem 21.6). Let U \subset E be closed. then ϕ r is a CFP for every r > 0. 19.1 Series connection of six resistors, is called the Radon-Nikodym Proof One direction is trivial. n=0 Since { $\sigma 0 = \infty$ } = A, we have P[$\sigma 0 < \infty$] = 1. i=1 By Theorem 2.5, the family ((pN)c, p \in P) is also independent, whence $(p \in P) = 1 - p - s$. Then $E[X1 \ T] = E[X1 \ E]$ a.s. and $1 \ n \to \infty \ Xi - \to E[X1 \ E]$ n n a.s. and in L1. As shown above, β is subadditive; thus $\alpha(C) \le \beta$ n Ai $i=1 \le \infty$ $\beta(Ai)$., XtN in P-probability. Let $J := \{i \in J : P[Bi] > 0\}$. In this case, $gh \ X_{\infty}$ is a version of the Radon-Nikodym derivative $d\mu$. For $p = \infty$, note that $|E[X|F]| \le E[X_{\infty} F] = X_{\infty}$. Chapter 16 Infinitely Divisible Distributions For every n, the normal distribution $N\mu,\sigma 2$ is the nth convolution power of a probability measure (namely, of $N\mu/n,\sigma 2/n$). (For the "a.s." notation see Definition 1.68.) Lemma 5.18 For $n \in N$, define $Yn := Xn 1\{|Xn| \le n\}$ and $Tn = Y1 + \cdots + Yn$. Let $Gn : [0, 1] \rightarrow R$, $t \rightarrow n-1/2$ ni=1 1[0,t] (F (Xi)) -t and $Mn := Gn \infty$ Consider now the general case. Concluding, X := F - 1 is the random variable that we wanted to construct. In Theorem 17.8 we saw that, for the semigroup of kernels (pn)n \in N, there exists a unique discrete Markov chain whose transition probabilities are given by p., $An \in Bb$ (E) pairwise disjoint. $n \to \infty$ (i) If P = w-lim Pn, then $\phi n \to \phi$ uniformly or compact sets. 1{Xn-m =a} . n $\rightarrow \infty$ As $\varepsilon > 0$ was arbitrary, we infer that fn $\rightarrow \rightarrow$ f uniformly distributed on [0, 1]. Proof This theorem was first proved by Aizenman, Kesten and Newman [2, 3]. As a consequence, we get uniqueness of the solution of the Dirichlet problem. Case 1: p = 1. n $\rightarrow \infty$ (8.7) 8.2 Conditional Expectations 197 Proof (i) The right-hand side is F -measurable; hence, for $A \in F$, *) **) E 1A $\lambda E[X | F] + E[Y | F] = \lambda E[1A X] + E[1A Y]$. Denote by $Sn = 2(X1 + . Now M := f(x0) + F \cdot X$ is a martingale by Theorem 9.39 since (and since F is predictable n 1 F is predictable. Thus we start with a short overview of some topological definitions and theorems. This implies $f + d\mu - f d\mu = g - d\mu = g d\mu$. is the classical Pólya's urn model. Hence we can define the expectation of X with respect to P[·|A]. Choose the partition P = {[0, 1/2), [1/2, 1]}. For \sigma-finite measures, the corresponding statement does not hold in this generality as we saw in Example 1.58(iv). Reflection Find an example that shows that without the tightness assumption, we need not have F $(-\infty) = 0$ nor F $(\infty) = 1.4$ We come to a first application of Prohorov's theorem., k + 1 = k+1 P[Xm, l = 0 = (1 - p)k+1. In this case, the probability that at time 2n all coordinates are zero would be the Dth power of the probability that the first coordinate is zero. Hence $\{\tau = s\} \cap Xs - 1$ (A) \in Fs \subset Ft for all $s \leq t$. (i) Normal distribution: Nµ1 , $\mu 2 \in R \ 1 \ 2 \ 1 \ 2$ and $\sigma 12$, $\sigma 2 = N\mu 1 + \mu 2$, $\sigma 2 + \sigma 2$ for all $\mu 1$, $\mu 2 \in R \ 1 \ 2 \ 1 \ 2$ and $\sigma 12$, $\sigma 2 > 0$. Hence averaging over one realization of many random variables is equivalent to averaging over all possible realizations of one random variable. , xn) for $x \in E$ n and for $x \in E$ N . . + ck = 1, then Y is called the moving average of X (with weights c0, . This coupling shows that bn, p1 \leq st bn, p2 . n $\rightarrow \infty$ 3 (ii) f dµn \rightarrow f dµ for all bounded Lipschitz continuous f. In this case, the derivative is ϕ (x) = D + $\phi(x)$. Let such that $\mu(W n n n \infty n) \leq \mu(Cn) + \epsilon 2 - n - 1 \leq \mu(B) + \epsilon$. Then n n (' tl 1Al (Y1) = 1 + $\nu(A)$) expression of X (with weights c0, . This coupling shows that bn, p2 = n + $\phi(x)$. tl - 1, $\psi(t) := E \exp i l = 1$ t $\in Rn$, l = 1 is the characteristic function of (1A1 (Y1), ..., $\omega n0$]] = (1 - p)n. In the general case, write X = X + - X - and Y = Y + -Y - and exploit the linearity of the conditional expectation. If Σ is finite or is a bounded subset of an Rd, then by symmetry, typically λ is the uniform distribution on Σ . (ii) The map $\kappa : E \times I$ $B(E) \otimes I \rightarrow [0, 1], (x, B) \rightarrow Px [X \in B]$ is a stochastic kernel. Denote by LipK (E; F) the space of Lipschitz continuous functions on E. Possibly all martingales Y? That is, $\{Y \in Am\} = \{\tau (Y) \in Am\}$. $n \rightarrow \infty$ a (4.6) $n \rightarrow \infty$ Theorem 4.23 (Riemann integral and Lebesgue integral) Let $f: I \rightarrow R$ be Riemann integrable on I = [a, b]. Example 20.36 (Rotation) We come back to the rotation of Example 20.9. Let $\Omega = [0, 1)$, $A = B(\Omega)$, $P = \lambda$ the Lebesgue measure, $r \in (0, 1)$ and $\tau r (x) = x + r \pmod{1}$. $\max(X1, \mu) = e^{-(\mu+\lambda)} n! m = 0$ Hence Poi $\mu + \lambda$. Clearly, we have fn (0) = 0 for all $n \in N$. If μ is a probability measure and if every κ is stochastic, then k=0 μ is a probability measure. However, the qualitative behaviour will be quite different. Then (Xi) i \in I is exchangeable. Lower semicontinuity follows from the monotone convergence theorem (Theorem 4.20). Fix an arbitrary $x0 \in E0 \cap E 0$. & Example 9.30 Let Y1, Evidently, the two-dimensional integer lattice is isomorphic to its dual graph., gm) starting and ending in some point $g0 = gm = u \in UL$, then $(g1, g2, With (i), the interpretation is clear. Consider the generalized version of Pólya's urn model (Xn) n \in N0$ with rk = r and sk = s for all $k \in N$. If $E[X] \in \partial I$, then X = E[X] a.s.; hence $E[\phi(X)] = \Phi(E[X])$. \blacklozenge Remark 4.25 An improperly Riemann integrable. The implication (i) \Rightarrow (ii) was shown in Theorem 18.12. Definition 17.58 Let $\mu 1$, $\mu 2 \in M1$ (Rd). One possibility is to show the claim first for measures on Rd. 4 19.2 Reversible Markov Chains Definition 19.8 The Markov chain X is called reversible with respect to the measure π if $\pi(\{x\})$ p(x, y) = $\pi(\{y\})$ p(y, x) for all x, y \in E. 15.2 Q-Q-plots for S100 abscissa shows the quantiles of the standard normal distribution. An (open) path (of
length n) in this subgraph is a sequence $\pi = (x0, x1, .)$ Hence X0 + Mn $\circ \tau \ge X0 + Sk \circ \tau = Sk+1$. That is, each point in T has exactly three neighbours. Without exaggeration, it can be stated that Brownian motion is the central object of probability theory. 2 ε n σ ε n By Slutzky's theorem (Theorem 13.18), we thus have convergence of the finitedimensional distributions to the Wiener measure PW : $n \rightarrow \infty$ PS⁻ $n \Rightarrow$ PW . Let π be the invariant distribution of X. $x \rightarrow D - \phi(x)$ is right continuous and $x \rightarrow D + \phi(x)$ is right continuous. If Ω is at most countably infinite and if $A = 2\Omega$, then the measurable space ($\Omega, 2\Omega$) is called discrete. Definition 7.20 For f, $g \in L2(\mu)$, define)f, $g^* := fg \ d\mu$. Note that Re(z) = (z + z)/2and Im(z) = (z - z)/2i imply cos(x) = eix + e - ix 2 and sin(x) = eix - e - ix 2i for all $x \in \mathbb{R}$. Yn) is measurable with respect to $\sigma(X1)$. For $x \in \mathbb{E}$, define (see Corollary 14.46) on (Ω , A) the probability measure Px such that, for finitely many time points 0 = t0 < t1 < .i, i = 0, . Proof By Theorem 6.19, there exists a monotone increasing convex function f with the property that $f(x)/x \to \infty$, $x \to \infty$ and $L := supi \in I E[f(|Xi|)] < \infty$. Hence we have) * P Nti – Nti–1 = ki for i = 1, . Just before each gamble we decide how much money we bet. For example, we want to compute the distribution of the functional F (X) := supt $\in [0,1] Xt$. (ii) τU is an F+ -stopping time but in general (even for continuous X) is not an F-stopping time. This clearly implies (21.30) with $\delta = (\eta/K)1/\gamma$. In this case, we have the Fourier inversion formula, f (x) = (2\pi)-d e-i)t, x* $\phi\mu$ (t) $\lambda(dt)$. z e δb , t Similarly, z e #c,t (vi) (Exponential distribution) This follows from (v) since exp $\theta = \Gamma \theta$, 1. (Compare [32, Exercise 2.1.24], see also [43, Section II.7].) Varadhan's lemma has various applications in statistical physics. 327 327 336 344 349 356 365 16 Infinitely Divisible Distributions . In particular, $B(Rd) = B(R) \otimes d$ for $i \in I \ d \in N$. To this end, we will need (21.5). Then $A = \{1, 2, 3\}$. Let $Ck \subset E \ k$ be measurable with $Ak = \{(X1, ..., f \ is \ p \ q \ twice \ continuously \ differentiable \ in (0, \infty) \ with \ derivatives \ f \ (x) = x \ p-1 - y \ and \ begin{subarray}{c} application \ begin{$ f(x) = (p-1)x p-2. Theorem 15.30 (Bochner) A continuous function of a probability distribution on Rd if and only if ϕ is positive semidefinite and $\phi(0) = 1$., $Mn,m = km \} \cap \{L = n\}$) * = P Mn, 1 = k1, . Hence, for $m \ge n \ge n0$, we have $f d\mu^n n = \lim_{n \to \infty} f 1W n d\mu klm = \lim_{n \to \infty} f 1W n d\mu klm$ f 1W n dµklm = lim $l \to \infty$ f 1W n dµklm f 1W n dµklm f 1W n dµklm = lim $l \to \infty$ f 1W n dµklm = lim $l \to \infty$ f 1W n dµklm f 1W n d $m d\mu klm = and thus f d \mu^n = lim m \rightarrow \infty f d\mu kmm$. Proof Let p be the transition matrix of X. k Example 18.19 Let $r \in (0, 1)$ and $N \in N$, $N \ge 2$. This probability should equal one. The frequencies for other languages can be found easily, e.g., at Wikipedia. Remark 17.26 The condition (17.16) cannot be dropped easily, as the following example shows. Then ν F 0 μ F (since F F C C in F there are fewer μ -null sets); hence the Radon-Nikodym derivative fF := (a) $d\nu$ F $d\mu$ F exists. Consider the set G \subset A1 \otimes A2 such that A \in G if and only if (14.6) and (14.7) hold for f = 1A. We show only the existence of a decomposition. fdd $n \rightarrow \infty$ (ii) Pn $\rightarrow \rightarrow$ P weakly. Further, for every $\mu \in F$, $\infty \quad \mu$ (A) $c \leq \mu(Ac) \leq \mu(Acn,N) \leq \epsilon$. (23.13) $\mu(\{x\})$ Since $\mu(\{x\}) > 0$ for all $x \in \Sigma$, the integrand ν -a.s. is finite and hence the integral also is finite. (i) Let $X \sim N\mu,\sigma 2$ and let $a \in R \setminus \{0\}$ and $b \in R$. .) is independent (but not necessarily independent of each other). The nth and the (n + 1)th superconductors are connected by 4(2n + 1) edges. For any $x \in E$, there exists a relatively compact neighborhood Bx x., An $\in B(Rd)$. b-a a, b a, b := lim Manifestly, By assumption, we $n \rightarrow \infty$ Un exists. The sets $A \in A$ are called measurable sets. (iii) (Linearity) If α , $\beta \in [0, \infty]$, then $(\alpha f + \beta g) d\mu = \alpha f d\mu + \beta g d\mu$, where we use the convention $\infty \cdot 0 := 0$. Further, let Rn = #(Rd). $\{S1, . < tnn = b\}$ that get finer and finer. Since $C1 \cap C2 = \emptyset$ and $C1 \cup C2 \subset A$, we get $\beta(A) \ge \alpha(C1 \cup C2) = \alpha(C1) + \alpha(C2) \ge \beta(A \cap Bc) + \mu * (A \cap B) - 2\epsilon$. Exercise 5.1.1 Let X be a nonnegative random variable with finite second moment. Recall the definition of a general coupling of two probability measures from Definition 17.54. be i.i.d. random variables and for $n \in N$, let Sn = X1 + . We start with a simple observation. Therefore, defining $S, T \in N X^{\tilde{c}}$ t (ω), $t \ge 0$, for $\omega \in \Omega \setminus \Omega \infty$, we get that $X^{\tilde{c}}$ is a locally Hölder continuous modification of X on $[0, \infty)$. (i) (ii) (iii) (iv) ϕ is convex. Theorem 14.31 (Kernels and convolution) Assume X1 , X2 , . (i) E = R and there is a $p \in [1, \infty)$ with $fn \in Lp(\mu)$ for all $n \in N$ and there is ∞ p an $f \in Lp(\mu)$ with $fn - fp < \infty$. $fd\mu^{n} m 292$ 13 Convergence of Measures This implies that for any measurable relatively compact set $A \subset E$, we have $\mu^{n} m (A) = \mu^{n} N(A)$ (A) for any $m \ge N(A)$. Define $\alpha(t) := E[Nt]$. If the jump rates are bounded, then the process can be constructed as a Markov chain at random times given by a Poisson clock. We come to the main theorem of this section. In this case, I (x, y) is called the flow from x to y and u(x) is is a Borel space. In either case, a probability measure on the product is uniquely determined by its values on cylinder sets. If $m \le 1$, then (Zn) $n \in \mathbb{N}$ converges a.s. to some random variable Z ∞ . with PTnr = PTns = expwn-1. (Of course, the chain with transition matrix $p(x, y) = \pi(y)$ converges to π , but this does not help a lot.) This method of producing $(approximately) \pi$ -distributed samples and using them to estimate expected values of functions of interest is called the Markov chain Monte Carlo method or, briefly, MCMC (see [15, 112, 119]). $n \rightarrow \infty$ (21.25), we have $Xt = X^{\tilde{}} t$ a.s. for all $t \in [0, 1]$. $) \in B = *$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$ ($Xn = x\}$ ($Xn = x\}$) and Xn = 1 ($Xn = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) ($Xn = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) ($Xn = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) ($Xn = x\}$ ($Xn = x\}$) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ ($Xn = x\}$) (Xn = x) (Xn = x) $E \pi 1A\epsilon 1 \{XN = x\}$ (Xn = x) (XnShow that, analogously to the Radon-Nikodym theorem, the following two statements are equivalent: (i) $\phi(A) = 0$ for all $A \in A$ with $\mu(A) = 0$.
\in DE are mutually disjoint., k. However, this is not a distribution function, as 1 does not converge to 0 for $x \to -\infty$. I (N) n = 1 Here we defined (n) $l := n(n - 1) \cdots (n - l + 1)$. By Lemma 13.5, there exists a compact set K with $\mu(K c) < \epsilon/2$. Then f (x) := G(x, a) is harmonic on E \ {a}: For x = a, we have pf (x) = p ∞ pn (x, a) = G(x, a). (2.16) Accordingly, let m = 2, 3, . (18.14) As all eigenvalues are real, the corresponding eigenvectors are given by xkn = 2 r 1 - r n/2 sin n π N, k = 1, . + T D, Yti := Z i i for i = 1, . j (10.5) Example 10.9 We want to generalize the preceding example further. C E be relatively compact open sets covering E. The investigation of product spaces and their σ -algebras is, however, postponed to Chap. Together with the topological considerations in Sect. n=1 Euler's prime number formula is a representation of the Riemann zeta function as an infinite product -1ζ (s) = 1 - p-s, (2.5) p \in P where P := {p \in N : p is prime}. \blacklozenge 456 18 Convergence of Markov Chains Example 10.19 with the probability of a gain r \in (0, 1). By the preceding example, we conclude the following theorem. This property of λ n is called inner regularity. Define $h = \infty 2 - n 1 + \mu(An) - 1 1An$. Example 12.21 Any family (Ai) i \in I of sub- σ -algebras of F is independent given F. Hence the variational problem for A* (x) admits a unique maximizer t * (x). Counterexample for semirings: Let $\Omega = \{\emptyset, \Omega, \{1\}, \{2, 3\}, \{4\}\}$ and $A2 = \{\emptyset, \Omega, \{1\}, \{2\}, \{3, 4\}\}$. As a simple conclusion of Lemma 14.44 and Theorem 14.45, we get the following statement that we formulate separately because it will play a central role later. are independent with distribution $\Xi \infty$ given $\Xi \infty$. \bigstar On the one hand, improperly Riemann integrable functions need not be Lebesgue integrable. (18.2) Lx, y is uniquely determined, and we have Lx, y + $Ly_z + Lz_x = 0 \pmod{d}$ for all x, y, z \in E. expectation (Theorem 8.14(ii)), for t > s, we have E[Zt Fs] \leq Xs , we have E[Zt Fs] \leq Xs and E[Zt F X2, (20.9) $x,y \in E \\$ Example 20.33 (Integer rotation) Consider the rotation of Example 20.8. Let $n \in N \setminus \{1\}$, E = Z/(n) and let P be the uniform distribution on Ω . For any $\omega \in \Omega$ and $n \in N$, let πn (ω) := n pXi (ω) i=1 be the probability that the observed sequence X1 (ω), (4.8) 0 3Proof Define 3 f = f ! and f = "f #., 6}, A = 2\Omega and P is the uniform distribution on Ω . Check that X(n) = Yn.* Show that the conditional distribution L X(1) - X(n), Inductively, we get $E[|Xt|] < \infty$ for all $t \in N0$. If X is aperiodic and positive recurrent with invariant distribution π , then we ; $n \rightarrow \infty$ have ; $L\mu[Xn] - \pi$; $T V \rightarrow 0$ for all $\mu \in M1$ (E). For any finite subset $J \subset I$, let $FJ := F(Xj)j \in J : RJ \rightarrow [0, 1], +, *$) Xj-1 ($-\infty$) x]. In this case, we can choose A = E or A = T. Hence, by Theorem 6.19, there exists a monotone increasing convex map $f: [0, \infty) \rightarrow [0, \infty)$ p with $f(x) x \rightarrow \infty$ and $C := E[f(|X0|)] < \infty$. 6} such that $A1 = A^{\tilde{}} 1 \times \{1, ..., 0\}$ (x1, ..., By Theorem 19.7, u is harmonic and can be written as '(ux1, A0 (x) = Ex 1{XTA \cup {x} = x1 } 01)* = Px Tx1 < TA0 for every x \in E (A0 \cup {x1}). If M \subset N, then denote by gcd(M) the greatest common divisor of all $n \in M$., n = 1 - n $e - \theta i x = 1 - exp - (\theta 1 + . Further, let W be the vector space of such ha, b. In fact, in Example 15.5, we saw that here the moments do not determine the$ distribution of X. 421.2 Construction and Path Properties 523 3 2.5 2 1.5 1 0.5 0 0.5 1 1.5 2 Fig. We perform the proof by induction on n. Proof "(ii) \Rightarrow (i)" For any $f \in V$, by definition of the inner product, the map $2.5 \times 1.5 \times 10.5 \times 10.$ \rightarrow)x, f * is linear. 26.3 Weak Uniqueness via Duality. (vi) The Poisson distribution is infinitely divisible with Poi= Poi * n λ/n . n=1 By Kolmogorov's 0-1 law (Theorem 2.37), T is P-trivial. B \in Pn+ $\phi(B) = \phi(Cn)$. For $\alpha \in [-1, 1]$, define probability densities fa on $(0, \infty)$ by fa $(x) = f(x) 1 + \alpha \sin(2\pi \log(x))$. Then $\mu(A) > 0$ and $(\nu - \epsilon \mu)(E) \ge 0$ for all $E \subset A$, $E \in A$. By Fatou's lemma (Theorem 4.21), we obtain $n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim \inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ ($\epsilon \epsilon c \le 1 - E \lim inf 1(A\epsilon n) c n \rightarrow \infty + , '$ (inf 1(A\epsilon n) c n \rightarrow \infty + , ' (inf 1(A\epsilon n) c n \rightarrow \infty + , ' (inf 1(A\epsilon n) c n \rightarrow \infty + , ' (inf 1(A\epsilon n) c n \rightarrow \infty + , ' (inf 1(A\epsilon n) c n \rightarrow The maximizer t * = t * (z) of the variational problem for $\Lambda *$ solves the equation $z = \Lambda (t *) = tanh(t *)$. (iv) The distribution of X1 can be chosen such that ϕ is differentiable at 0 but $E[|X1|] = \infty$. (ii) For any $A \in F$, we have E[X1A] = E[Y1A]. In fact, Sn = ni=1 (1 - Di). Proof " \Rightarrow " First assume μ is infinitely divisible. Indeed, without loss of generality, assume 0 is the left boundary of I and A := {E[X |F] = 0}. (i) I (αf) = α I (f). The full procedure is rather technical and does not allow for a smooth intuitive description. Example 12.5 (i) For x \in RN, define the nth arithmetic mean by an (x) = n1 ni=1 xi. However, the notational complications become overwhelming for n \geq 3, and the idea for general $n \in N$ becomes clear in the case n = 2. π The second largest modulus of an eigenvalue is $|\lambda n| = \sigma \cos N$ if n = 1 or n = N - 1. This implies ('(' $\phi n, l(t) - 1 \le E$ eit Xn, l - 1 1{ $|X| > \epsilon$ } . Choose C1 $\in C$ with C1 $\subset A \cap B c$ and $\alpha(C1) > \beta(A \cap B c) - \epsilon$. Finally, $P[N\epsilon \ge 2] = 1 - e - \alpha\epsilon - \alpha\epsilon$ $e - \alpha \varepsilon = f(0) - f(\alpha \varepsilon)$, where f(x) := e - x + xe - x. (ii) Since ϕ is convex, so is $x \to \phi(x) + .$ (i) What is the probability that the last person gets his or her reserved seat? By Theorem 13.16, $\lim n \to \infty$ if $\mu \circ \phi - 1 = \lim n \to \infty$ if $\mu \circ \phi - 1 = \lim n \to \infty$ if $\mu \circ \phi - 1 = \lim n \to \infty$ if
$\mu \circ \phi - 1 = \lim n \to \infty$ if $\mu \circ - 1 = \lim n \to \infty$ if $\mu \to \infty$ if approximates f uniformly on a suitable set C. \neq Exercise 15.1.2 Let d \in N and let μ be a finite measure on $[0, \infty)d$. Jensen's inequality can be extended to Rn. .) = σ (Y-n, Y-n-1, Y-n-2, . The question is a bit tricky since for every given A \in A, the expression P[A|X = x] is defined for almost all x only; that is, up to x in a null set that may, however, depend on A. However, a σ -finite measure on Z is not uniquely determined by the values on E: Let μ be the counting measure on Z and let $\nu = 2\mu$. Takeaways Mixing is a concept of independence stronger than ergodicity but weaker than stochastic independence. (In particular, continuous maps are lower semicontinuous. A Chapter 17 Markov Chains In spite of their simplicity, Markov processes with countable state space (and discrete time) are interesting mathematical objects with which a variety of realworld phenomena can be modeled. (12.7) Proof By Theorem 12.10, An (ϕ) = E[$\phi(X)$ En]. Y := n=1 (iii) In particular, if we let N ~ Poi λ in (ii), then ϕY (t) = exp($\lambda(\phi X (t) - 1)$). , y) \in R. In order for the measure extension to work, σ -additivity is decisive. + Xn for $n \in N0$. It is maximal (in fact, log(#E)) if p is the uniform distribution on E. Show that f - hp < ϵ . Let (E, d) be a metric space and let F \subset M ≤ 1 (E) be tight. 138 5 Moments and Laws of Large Numbers Theorem 5.30 (Rademacher-Menshov) Let X1, X2, Left side: below the critical temperature ($\beta > \beta c$); Right side: above the critical temperature. Theorem 18.11 Let p be the transition matrix of an irreducible, positive recurrent, aperiodic Markov chain on E. Hence X = (Xn) n \in N0 is a stationary real-valued stochastic process. 15.4 Characteristic Functions and Moments 355 (ii) Assume that for any $k \in N$ the limit $mk := \lim mk (Xn)$ $n \to \infty$ exists and is finite (note that finitely many of the mk (Xn) not $k \in N$ and a subsequence (Xnl) $l \in N$ such that $l \to \infty$ PXnl $- \to PX$ weakly. Let $f : Rd \to R$ be continuous with compact support and let $\varepsilon > 0$. (iii) Show the theorem of Fréchet-Shohat: If in (ii) the distribution of X is determined by its moments mk (X), $k \in N$ (see Corollary 15.33), then $n \rightarrow \infty$ PXn $- \rightarrow$ PX weakly. (ii) For this particular ψ , all the iterates are of a special form and can be computed explicitly. Indeed, $\{Ym \le k\} = k+1 \ \{Xm, l = 1\} \in \sigma$ (Xm, l, l = 1, . Clearly, $\{\tau \le t\} \subset A$ for all $t \in I$; hence $A \cap \{\tau \leq t\} = \{\tau \leq t\} \in Ft$. For $r \in [0, 1)$, define $\tau r : \Omega \to \Omega$ by $\tau r(x) = x + r - x + r! = x + r \pmod{1}$. Now fix $p \in [0, 1]$ and let X1, X2, Definition 21.4 (Path properties) Let $I \subset R$ and let X = (Xt, t \in I) be a stochastic process on some probability space (Ω, A, P) with values in a metric space (E, d). Here as the orthonormal basis of L2 ([0, 1]) we use b0 = 1 and $bn(x) = \sqrt{2} \cos(n\pi x)$ for $n \in N$. That is, the new state space is $E^{\sim} = E \cup \{\Delta\}$ and the transition matrix is p(x, y) = 0, $|| = \mu 2$, $p(x, \gamma) = 0$, $|| = \mu 2$, $p(x, \gamma) = 0$, $|| = \mu 2$. $p(x, \gamma) = 0$, $|| = \mu 2$, $p(x, \gamma) = 0$, $|| = \mu 2$. Proof Let $f \in L^{\infty}(\mu)$ and $p \in [1, \infty)$. n pk $\phi\mu k$. Since we will not need these statements in the following, we only refer to the standard literature (e.g., [174, Chapter VI.2] or [54, Theorem XV.3.3 and Equation (XV.3.8)]). Let $f : \Omega 1 \times \Omega 2 \rightarrow R$ be measurable with respect to A1 \otimes A2. Klenke, Probability Theory, Universitext, 53 54 2 Independence (i) Two events A and B should be independent, e.g., if A depends only on the outcome of the first roll and B depends only on the outcome of the section, we show the functional central limit theorem, which goes back to Donsker [35]. 12.3 De Finetti's Theorem In this section, we show the able families that was heuristically motivated at the end of Sect. 707 Chapter 1 Basic Measure Theory In this chapter, we introduce the classes of sets that allow for a systematic treatment of events and random observations in the framework of probability theory. be maps $E \to n \to \infty R$ with fn $- \to$ pointwise. Then $\lim \phi Sn * (t) = e - t 2/2 n \rightarrow \infty$ for all $t \in \mathbb{R}$. (v) If (Xt)t $\in \mathbb{N}0$ is a supermartingale and $\mathbb{E}[XT] \ge \mathbb{E}[X0]$ for some $T \in \mathbb{N}0$, then (Xt)t $\in \{0, \dots, T\}$ is a martingale. Hence, we obtain) * $n \rightarrow \infty$) * Lx $\lambda 1 Z^{\tilde{c}} tn1 + \lambda 2 Z^{\tilde{c}} tn2 \rightarrow Lx \lambda 1 Yt1 + \lambda 2 Yt2$. The claim is immediate. Definition 21.33 Let P be the probability measure on $\Omega = \mathbb{C}([0, \infty))$ with respect to which the canonical process X is a Brownian motion. This implies that, for any $\alpha \in \mathbb{R}$, the set $\{f \leq \alpha\} \cap \{g = h\}$ is the union of a B(I). Definition 14.9 (Cylinder sets) For any $i \in \mathbb{I}$, let $\mathbb{E} i \subset \mathbb{A}$ be a subclass of the class of the clas measurable sets. Define $u(t) = \text{Re}(\phi(t))$. A random measure X with independent increments is called a Poisson point process (PPP) with intensity measure μ if, for any $A \in Bb$ (E), we have PX(A) = Poi $\mu(A)$. With a little effort it is possible to construct ϕ as a (random) additive set function. However, then fnkl $\in U$ for all but finitely many l, which yields a contradiction! Corollary 6.15 Let (E, d) be a separable complete metric space. k = 0 (iii) Let $p \in (0, 1]$ and $X : \Omega \to N0$ with P[X = n] = p(1 - p)n for any $n \in N0$. Reflection Typically, the distribution of W cannot be computed explicitly. Hence, let $p \in (1, \infty)$. The integral of an integrable function on the product space can be computed by successive integration (in arbitrary order) over the individual coordinates (Fubini's theorem)., $ck \in R$. Then the map If : $\Omega 1 \rightarrow [0, \infty]$, $\omega 1 \rightarrow f(\omega 1, \omega 2)$ is well-defined and A1 -measurable. Theorem 5.16 Let X1, X2, . Hence X is an F-martingale. We model the experiment on the probability space (Ω , A, P), where $\Omega = \{1, ..., \alpha 0\}$, $\omega 1 \rightarrow f(\omega 1, \omega 2)$ is well-defined and A1 -measurable. Theorem 5.16 Let X1, X2, . Hence X is an F-martingale. We model the experiment on the probability space (Ω , A, P), where $\Omega = \{1, ..., \alpha 0\}$, $\omega 1 \rightarrow f(\omega 1, \omega 2)$ is
well-defined and A1 -measurable. drops you see on the side walk. Hence, by Theorem 19.30, random walk on Z3 is also transient. 24.1 for a simulation of a Poisson point process on the unit square. The bars show the reflection principle for Brownian motion. 6.1). We thus get a random sequence (Xn)n \in N0 of states in {0, 1} Λ that represents the random evolution of X is analytic and the distribution of X is n uniquely) t |X| * determined by the moments E[X], $n \in N$. , Xk = xk, $tx1 = \infty'$ (= Px [X1 = x1, . Hence, in terms of the Markov chain notation, we have <math>E = ZD and p(x, y) = 1 2D, 0, if |x - y| = 1, else. Show that, for every $X \in L1(P)$, we have E[X|A0] = 1 $X \circ g$. Hence, for any $n \in N$, $f(x0) = (pA)n f(x0) = n pA(x0, y)f(y) \le m n(x) y \in SA 0$ with equality if and only if (y) = m for all $y \in SAn(x0)$. We define the conditional expectation of Y given X = x by $E[Y | X = x] := \phi(x)$, where ϕ is the function from (8.10) with Z = E[Y | X]. The transition matrix is $[p(x_1, x_2), p(y_1, y_2), q(x_1, y_1), (x_2, y_2)] = p(x_1, x_2) + p(y_1, y_2)$, $[p(x_1, x_2), p(y_1, y_2), q(x_1, y_1), (x_2, y_2)] = p(x_1, x_2) + p(y_1, y_2)$. follows from (vi)., $\omega n : [\omega 1, .$ Furthermore, $E[X(X - 1)] = n \ k(k - 1) \ P[X = k] \ k = 0 = n \ k = 0$ $n \ k \ k(k - 1) \ p(1 - p)n - k \ n \ n - 1 \ k - 1 = np \cdot (k - 1) \ (1 - p)(n - 1) - (k - 1) \ p(1 - p)(n - 2) - (k - 2) = n(n - 1)p2 \cdot n \ n - 2 \ k = 2 \ k - 2 \ pk - 2 \ (1 - p)(n - 2) - (k - 2) = n(n - 1)p2 \cdot F$ is called a filtration if $Fs \subset Ft$ for all s, $t \in I$ with $s \le t$., we have D(X1, X2, . Hence, 0.5 0.0 $0 \frac{1}{2} \frac{$ with $(H \cdot X)0 = 0$; hence clearly $E[(H \cdot X)T] = 0$. By the dominated convergence theorem (Corollary 6.26), the limiting function f (\cdot , x0) is in L1 (μ) and F (xn) – F (x0) lim = lim n \rightarrow \infty n – x0 gn (ω) $\mu(d\omega) = f(\omega, x0) \mu(d\omega) = f(\omega, x0) \mu(d\omega)$. variation process of M. \cap An) = n k=1 {i1,...,ik} C {1,...,n} (-1)k-1 μ (Ai1 \cup . Example 14.15 For i = 1, . & Exercise 15.3.2 Show that for any $\mu \in M1$ (R) with characteristic function ϕ , we have $\varepsilon \mu([-\delta, \delta]c) \leq C(1 - \operatorname{Re}(\phi(t)))$ dt. Now consider bond percolation on T with probability p. Theorem 17.39 If E is finite and X is irreducible, then X is recurrent. \bullet Reflection Check the statements of the preceding remark! \bullet Definition 15.8 For $\mu \in Mf(Rd)$, define the map $\phi\mu$: Rd $\rightarrow C$ by $\phi\mu(t) := ei)t, x^*\mu(dx)$. By Step 2, we have $qx \in C$. For $F \subset A$, show that $E[X | F] \in I$ a.s. Is this statement still true if we require only $X - \in L1$ (Ω, A, P) instead of $X \in L1$ (Ω, A, P) instead of $X \in L1$ (Ω, A, P)? Theorems of this type are also called invariance principles since the limiting distribution is the same for all distributions Yi with expectation 0 and the same variance. 477 2 n 12 edges n+1 4(2n + 1) edges Fig. Let $B \in Fn$ and $m \ge n$. By Theorem 5.4, Y1 · Y2 is integrable. " \leq " This follows from Hölder's inequality. Exercise 6.1.3 (Egorov's theorem (1911)) Let (Ω, A, μ) be a finite measure space and let f1, f2, .17.4, in particular, Definitions 17.29 and 17.34. In $\rightarrow \infty$ Exercise 6.1.3 (Egorov's theorem (1911)) Let (Ω, A, μ) be a finite measure space and let f1, f2, .17.4, in particular, Definitions 17.29 and 17.34. Theorem 11.14. Show that for any $\varepsilon > 0$, there is a continuous function $h : R \to R$ such that $f - hp < \varepsilon$. In practice, it is often not possible to check if a map X is measurable. Takeaways In this section, we have compiled a wish list of the properties that a probability assignment should have: σ additivity and normalization (Definition 1.28). Hence, clearly, $N \rightarrow \infty$ d(fN, g) $\rightarrow 0$, and thus d is complete. If $A \in I$, then 1A is I-measurable and hence P-a.s. equals either 0 or 1. A map $h : \mathbb{R} \rightarrow \mathbb{R}$ is called a step function if there exist $n \in N$ and numbers to < t1 < . By virtue of the Borel-Cantelli lemma, show that 1 lim sup $Xn = n \rightarrow \infty n 0$ a.s., if E[X1] < . ∞ , ∞ a.s., if E[X1] = ∞ . Define 2 i=1 i=1 A := Nn ∞ B1/n xin. Klenke, Probability Theory, Universitext, 367 368 16 Infinitely Divisible Distributions choose a real-valued CFP ϕ for which $|\phi| = \phi$ is also a CFP (see Examples 15.17 and 15.18). Hence we have Ai \in I for i = 0, . (b) For $\alpha \ge 0$, we have $\alpha f d\mu = \alpha f + d\mu - \alpha f - d\mu = \alpha f + d\mu - \alpha f - d\mu = \alpha$ f dµ. Theorem 1.81 (Measurability on a generator) Let E ⊂ A be a class of A measurable sets. 334 15 Characteristic Functions and the Central Limit Theorem Exercise 15.1.1 Show that, in the Stone-Weierstraß theorem, compactness of E is essential. (ii) Consider an urn with N balls, M of which are black. , n – 1, changes its value exactly twice. n! (3.4) Hence X + Y has probability generating function ψ Poi λ (z) · ψ Poi μ (z) = $e\lambda(z-1) = \psi$ Poi $\lambda+\mu$ (z). By symmetry, we have C(x, y) = C(y, x) for all x and y. If r is irrational, this implies cn = 0 for n = 0, and thus f is almost surely constant. Theorem 1.53 (Extension theorem for measures) Let A be a semiring and let μ : A \rightarrow [0, ∞] be an additive, σ -subadditive and σ -finite set function with $\mu(\emptyset) = 0$. Lemma 19.24 For any decreasing sequence An0 $\downarrow \emptyset$ such that $|E \setminus An0| < \infty$ and $x1 \in An0$ for all $n \in N$, we have Ceff $(x1 \leftrightarrow \infty) = \lim Ceff(x1 \leftrightarrow An0)$. If $\theta(p) > 0$, then $\psi(p) \ge \theta(p)$ implies $\psi(p) = 1$. If $f \in C$, then by assumption Re(f) = $(f + f^{-})/2$ and Im(f) = $(f - f^{-})/2$ are in C. (iii) Conclude that a suitable continuous version of Gn converges weakly to B. Other information that may be included if you're looking up a business is the company profile and a link to the company profile and a link to the company profile and a link to the company website. The situation changes when the numbers rk grow quickly as $k \rightarrow \infty$., $\omega k] \subset Cn, in$. Proof Let $x \in V$ and $c := \inf\{x - w : w \in W\}$. Let $\tau 0 = 0$ and inductively define $\tau n+1 = \tau n + \sigma \tau n$ for $n \in N0$. Theorem 7.3 says that this norm is complete (i.e., every Cauchy sequence converges). be i.i.d. random variables with values in Σ and with distribution PY1 = ν . 1 By "countable" we always mean either finite or countably infinite. Theorem 2.21 A family (Xi)i \in I of real random variables is independent if and only if, for every finite $J \in R$ and every $x = (x_j) \in F$ and every $x = (x_j) \in F$. Then C is dense in Cb (E; K) with respect to f = 0 and U3n (f) = 1 for all $n \in N$. Fix $\delta > 0$ and choose $N \in N$ large enough that $\mu(B \setminus AN) < \delta$. Use Exercise 1.5.4 to show that $(x, y) \rightarrow F(x) \wedge G(y)$ is a distribution function on R2. , Y1 + Y2 + . Then C is dense in Cb (E; K) with respect to the supremum norm. Let 0 1, if $un \in S$, Xn = 0, else. 9.1 Processes, Filtrations, Stopping Times.. For example, for nearest neighbour random walk on the integers, every state has period d = 2. Theorem 9.35 Let X be a martingale and let $\phi: R \to R$ be a convex function. The Rn that we introduced in the explicit construction are given by Rn (x) := x + Zn $P(x) = \sqrt{x} 2\pi x \ln particular, \infty$)* P sup B[n,n+1] - Bn > n \epsilon < \infty for all A, B \in A, P, \tau) is ergodic if and only if, for all A, B \in A, P, \tau) is ergodic if and only if, for all A, B \in A, A \in A, B \in A, A \in A, A \in A, A \in A, B \in A, A \i complete metric on E that induces τ . Takeaways Consider an event that is described by the values of infinitely many random variables. ***** Exercise 21.4.5 Show the statement of Remark 21.23. In order to obtain the density of ν_a with respect to μ , we define $f := g \ 1\Omega \setminus E$. (5.16) k=1 ('Hence P Nt = lim Ntn = 1. To be more specific, the question is: In the long run, will there be a consensus of all individuals or will competing opinions persist? We have to show that, for every bounded measurable $F : \mathbb{R}[0,\infty) \to \mathbb{R}$, we have:) * Ex F (Bt + τ)t ≥ 0 F $\tau = EB\tau$ [F (B)]. / Proof Let $\Omega = \mathbb{R}n$ and $A = B(\mathbb{R}n)$. be subsets of Ω . n=1 Thus A is $\sigma \cdot \cup -closed$. Let A be the ring of finite unions of intervals (a, b] $\subset \mathbb{R}$. Since it Xn.l ϕ n.l (t) -1 = -1] E[e] $= 1 = 1 \le kn^*$) E eit Xn.l -i tXn.l -i tXn.l -i tXn.l -i t $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l]
$= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \ge 2$ E[Xn.l] $= 1 \le kn^2$ t $= 1 \ge 2$ E[Xn.l] $= 1 \ge 2$ E[Xn.l] = 1 \ge 2 E[Xn.l] $= 1 \ge 2$ E[Xn.l] $= 1 \ge 2$ E[Xn.l] $= 1 \ge 2$ E[Xn.l] = 1 \ge 2 E[Xn.l] $= 1 \ge 2$ E[Xn.l] = 1 \ge 2 E[Xn.l] $= 1 \ge 2$ E[Xn.l] = 1 \ge 2 E[Xn.l E[Xn,l] = 0, kn kn Clearly, pc is irreducible and aperiodic. $n \rightarrow \infty$ 16.1 Lévy-Khinchin Formula 371 Inductively, we get $|\phi(t)| \ge 2-(4)$ for $|t| \le 2k \in .34$ 1 Basic Measure space. As $\{x\} \in B(Rn)$ for a measure space) Let (Ω, A, μ) be a σ -finite measure space. As $\{x\} \in B(Rn)$ for a measure space. As $\{x\} \in B(Rn)$ for $|t| \le 2k \in .34$ 1 Basic Measure space) Let (Ω, A, μ) be a σ -finite measure space. As $\{x\} \in B(Rn)$ for a measure space. As $\{x\} \in B(Rn)$ for a measure space of $(1 \le 2k \in .34)$ for $|t| \le 2k \in .34$ 1 Basic Measure space). a content on A, then by Lemma 1.31, for A, $B \in E$ such that $\mu(A)$, $\mu(B) < \infty$, we have $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$. Then, for every probability $n \rightarrow \infty$ n measure $\mu = \alpha 1 \mu 1 + .$ Since all continuous for every $\gamma \in (0, n-1, 2n)$ and every $n \ge 2$ and hence for every $\gamma \in (0, 12)$. Theorem 1.78 (Generated σ -algebra) Let (Ω , A) be a measurable space and let Ω be a nonempty set. Then P - QT V = inf $\phi((E \times E) \setminus D)$: $\phi \in K(P, Q)$. We thus obtain the Metropolis transition matrix ' ($\int 11 | 1 \wedge \exp 2\beta (1 -)$, if y = x i for some $i \in \Lambda$, $\{x(j) = x(i)\} | 2 | \#\Lambda j : j \sim i p(x, y) = 1 - i \in \Lambda p(x, x i)$, if x = y, $| | [0, else. Hence (Pn) n \in N P = 0$ is tight. Many people have helped in correcting errors or improving the exposition by asking questions and I thank all of them. Example 9.40 (Petersburg game) We continue Example 9.40 (Petersburg g Functions and the Central Limit Theorem For $m \in \mathbb{Z}d$ define $x \to \exp i$) $\pi m/K$, x^* . For $T \ge 0$, let $|X| * T = \sup |Xt|$. Let ξn (X) be the empirical measure of X1, . \bullet In fact, the condition $\Lambda(t) < \infty$ for all $t \in \mathbb{R}$ can be dropped. \bullet Example 9.5 The Poisson process $X = (Xt) t \ge 0$ with intensity $\alpha > 0$ (see Sect. 15.2 Characteristic Functions: Examples. Further, let $f0 \equiv 1$. (2.13) 2.4 Example: Percolation 75 Lemma 2.40 Let x, $y \in Zd$. l=1 Thus X(A1), E (Xt - Xs) 2n = Et - sX1 Now let $n \ge 2$ and $y \in (0, n-12n)$. We single out an arbitrary point of T and name it 0. Cw (i) i + 1 and Hence the transition probabilities pw are indeed described by the Cw. As a first step, we enumerate $E = \{e1, ..., Ai \text{ for all } n \in N\}$. 3 3 Proof Let $\mu 1$, $\mu 2 \in Mf(E)$ with $g d\mu 1 = g d\mu 2$ for all $g \in C$. = $\lim n \to \infty n$ Therefore, $\lim n f \log Pn(x - \varepsilon, \infty) - Pn[x + \varepsilon, \infty) n \to \infty n$ 1 = $\lim n f \log Pn(x - \varepsilon, \infty) \ge -I(x)$. Let τ be the product topology on $\Omega = \times \Omega i$ and $i \in I B = \sigma(\tau)$. Hence $p(x, y) = p(x, y) 1x \in E \setminus A$. As T is a Poisson process with rate 1, (XTt) t ≥ 0 is also a Markov process with Q-matrix q. Then (fg) \in L1 (μ) and 1 p + 1 q = 1 and f \in fg1 \leq f p \cdot gq. This generalizes the Lebesgue integral that can be found in textbooks on calculus. 11.2 Martingale Convergence Theorems 251 This implies M $\infty \in \{0, Ld\}$. Similarly, we get (7.7) for all measurable h ≥ 0 . Example 23.14 Let $\Sigma = \{-1, 1\}$ and let $\mu = 12 \delta - 1 + 12 \delta 1$ be the uniform distribution on Σ . Exercise 6.3.1 Let X be a random variable on (Ω , A, P) and let $\mu = 12 \delta - 1 + 12 \delta 1$ be the uniform distribution on Σ . Exercise 6.3.1 Let X be a random variable on (Ω , A, P) and let $\mu = 12 \delta - 1 + 12 \delta 1$ be the uniform distribution on Σ . 15.33). \in Mf (Rd) and let p1 , p2 , . . , Yn) be independent and Berx -distributed. The rate function that shows up here is the analogue to the free energy of thermodynamics. , jn }, we have $\kappa(x, \cdot) \circ XJ - 1 n - 1 = \delta x \otimes (14.16) \kappa_j k_j k + 1$. Then $1 * P[A n] = \infty$; however, P[A] = P[A1] = 6. 18.2 Here N(8, 8) = {6, 10, 12, 14, 16, . If x > K is a point of continuity of F, then $\lim \sup k \to \infty$ Fnk (∞) $\leq \lim \sup k \to \infty$ Fnk (x) + $\varepsilon = F(x) + \varepsilon \leq F(\infty) + \varepsilon$. (5.8) n=n0 The aim is to employ Lemma 5.20 to refine the estimate (5.7) for (Yn)n \in N and (Tn)n \in N. Takeaways Loosely speaking, a family of functions is uniformly integrable if the main contributions to the integrals of those functions do not come from extremely large values of the functions. (iii) If $X \in L2$ (P), then X is called square integrable and ' (Var[X] := E X2 - E[X]2 © The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. Hence equation (13.12) holds. For a countable product of Polish spaces, the Borel σ -algebra of the product is the product of the Borel σ -algebras. \bullet Example 15.18 Define the function ψ : R \rightarrow [0, 1] for t \in [$-\pi/2$, $\pi/2$] by $\psi(t) = 1 - 2|t|/\pi$., d -1. \bullet Takeaways A random variable x is a measurable map from a probability space to some measurable map from a probability space to some measurable x is a measurable x is a measurable map from a probability space to some measurable x is a measurable map from a probability space to some measurable x is a measurable x is a measurable map from a probability space to some measurable x is a measurabl $= 1\{|x-y|=1\}$. independent real random variables with $E[|Xn|] = \infty$ for all $n \in N$ but such that X1 + . i=1 Proof Let $z \in [0, 1)$ and write $\psi X1(z) \psi X2(z) = P[X1 = n] z P[X2 = n] = 0$ $n = 0 = \infty$ $n = 0 = \infty$ n = 0 $= \psi X1 + X2$ (z). Every terminal event is symmetric. For instance, let A be the event where the sum of the two rolls is odd, $A = (\omega 1, \omega 2) \in \Omega$: $\omega 1 \in \{1, 2, 3\}$. $\in E$ be a sequence such that $\infty \mu(\Omega n) < \infty n = 1 \Omega n = \Omega$ and c for all $n \in N$. "(ii) \Rightarrow (i)" Let $f \in Lip1$ (R; [0, 1]). (Recall that Ex denotes the expectation for X if X0 = x.) Due to the symmetry of Yi, we have $[1 \ge 1, 1] = 0$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$ and $x \ge a$, if $m \le n$. Let $x, y \in E$ with x = y and deduce by induction that $\pi(\{y\}) = 1$. C Cb (E). Let A = A - \cap A + . 4.1 Construction and Simple Properties 101 (c) We have + (-f) dµ - (-f) dµ = = f - dµ - (-f) - dµ f + dµ = - f dµ. n. Now replace F by the smaller filtration G = (Gn) n ∈ -N that is defined by G-n = σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By the Bolzano-Weierstraß (Fn (q1))n ∈ N has a convergent subsequence Fn1 (q1) + σ (Y-n, Xn+1, Xn+2, . By) $k \in \mathbb{N}$. \bullet Example 1.86 Let $d(x, y) = x - y^2$ be the usual Euclidean distance on Rn and let B(Rn, d) = B(Rn) be the Borel σ -algebra with respect to the topology generated by d. We denote by $q := \lim P[Zn = 0] n \rightarrow \infty$ the extinction probability; that is, the probability that the population will eventually die out. " = " Now let X be i.i.d. given A for a suitable
σ -algebra $A \subset F$. Originally we were also interested in the situation where Y takes values in Rn or in even more general space. < an and 1 = y0 > y1 > . Again it suffices to assume that f is an indicator function of the type f (x) = 1B1 × ···× Bn+1 (xt1) (A, C) = 1 Hence Rw,eff $(0 \leftrightarrow \infty) = 1$ 1 R $\omega - + 1$ R $\omega - + 1$ R $\omega + - < \infty$ or R + $< \infty$. This implies $\mu(An) - \mu(A) = n \rightarrow \infty$ $\mu(Bn) - \rightarrow 0$. Takeaways A Galton-Watson branching process (Zn) with mean offspring number m > 1 has a positive chance to survive and in this case grows indefinitely. N, define Fn := $\sigma(X1)$. Show that for any $k \in N \cap (0, r)$ and $s \in (0, r)$ we have Ms $(X) < \infty$ as well as $l \to \infty$ Ms (Xnl) $- \to$ Ms (X) and $l \to \infty$ mk (Xnl) $- \to$ mk (X). μ is σ -subadditive. Show that $\mu \varepsilon := N0, \varepsilon$ satisfies an LDP with good rate function I (x) = x 2 /2. To put it differently, the first derivative ϕ of a convex function is a monotone increasing function. , n} $\geq t \leq Var[Sn]$. (iii) In particular, if $p \geq 1$ and $E[|Xt|p] < \infty$ for all $t \in I$, then

(|Xt|p) t \in I is a submartingale. (ii) In particular, to any stochastic matrix p, there corresponds a unique discrete Markov chain X with transition probabilities p. Thus H \cdot X is a martingale. 7.4. At first reading, some readers might wish to skip some of the more analytic parts of this chapter. "(ii) \Rightarrow (iv)" It is enough to show that for any $\varepsilon > 0$, there is a compact set $K \subset E$ with lim supn $\rightarrow \infty$ µn ($E \setminus K$) $\leq \varepsilon$. 8.2 Conditional Expectations ... Then *) F (λ) = E $e - \lambda X$ is infinitely often differentiable in (0, ∞). Note that the order of quantifiers is subtle. Γ (r) Γ (s) Then β r,s is called the Beta distribution with parameters r and s. \in A and $A \subset$ there exists an $N \in N$ such that ∞ n=1 An . We can model this on a probability space (Ω , A, P) where $\Omega = \{-1, 1\}$ N, A = (2{-1,1}) \otimes N is the σ -algebra generated by the cylinder sets [$\omega 1$, . (7.10) n=1 The set of all signed measures will be denoted by M± = M± (Ω , A). Furthermore, due to the lack of memory of the exponential distribution (see Exercise 8.1.1), P[Xt + s ≥ n + 1]Xt = n] = P[Sn+1 ≤ t + s |Sn ≤ t, Sn+1 > c + s |Sn < t, Sn+1 > c + s |Sn < t, Sn+1 > c + s |Sn t] = P[Tn \leq s + t - Sn |Sn \leq t, Tn > t - Sn] = P[Tn \leq s] = 1 - exp(-n2 s). r-1 -z dz \leq cr e-c (1 + t 2)r/2 - \rightarrow 0 for c $\rightarrow \infty$. Klenke, Probability Theory, Universitext, 95 96 4 The Integral m Lemma 4.1 If f = i=1 α i 1Ai and f representations of f \in E+, then m n α i μ (Ai) = = n j = 1 β j 1Bj are two normal β j μ (Bj). If X is a martingale, then equality holds in each case. Define C := max C(t1), . Show that, for any $\varepsilon > 0$, the distribution PX is characterized by the values mX (s) (respectively mX (-s)), s $\in [0, \varepsilon]$. This implies Pn [X $\in A$] $\in \{0, 1\}$. The second step is to check that $\mu *$ is a measure at least on σ (E). \blacklozenge Example 1.75 (Uniform distribution) Let A $\in B(Rn)$ be a measurable set with ndimensional Lebesgue measure λn (A) \in (0, ∞). (ii) As X is irreducible, by Theorem 17.52, we have $\pi(\{x\}) > 0$ for every $x \in E$. Furthermore, f, $g \in G$ implies f $v \in G$. A stochastic matrix is essentially a stochastic matrix is finite market is thus arbitrage-free if and only if there exists an equivalent martingale (to be defined below). We have $\mu p = \mu$. k=0 Here, for $r \in (0, \infty)$ and $p \in (0, 1]$, $-br, p \propto -r(-1)k pr(1-p)k \delta k = k$ (3.7) k=0 is the negative binomial distribution with parameters r and p. be independent random variables with PXk = CPoi($\nu \mid$) for k = 0, 1, . $x,y \in E$ n $\rightarrow \infty$ By Theorem 18.13, we have $pn-N(x, y) \rightarrow \pi(\{y\})$ for all $x, y \in E$. with Var[Xn] = 1 for all $n \in N$ such that n -1 Xk = ∞ almost surely. A family $\nu = (\nu t : t \in I)$ of probability distributions on Rd is called a convolution semigroup if $\nu s + t = \nu s * \nu t$ holds for all $s, t \in I$. We will use it in many places. Example 1.56 (Lebesgue-Stieltjes measure) Let $\Omega = \mathbb{R}$ and $A = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$., whose square is a divisor of p) and let $q \in \{2, 3, .$ Hence $E[XT \mid A] = E[XT \mid A] = E[XT \mid A] = E[XT \mid A] = E[XT \mid A]$. change the probability. 15.1 Separating Classes of Functions Let (E, d) be a metric space with Borel σ -algebra E = B(E). Here the values + ∞ and $-\infty$, respectively, are possible. A map F : Rd \rightarrow R is called monotone increasing if F (x) \leq F (y) whenever $x \leq y$. We define hn (P, τ ; P) = $-P[A] \log(P[A])$. 9.4 Discrete Martingale Representation Theorem and the CRR Model By virtue of the stochastic integral, we have transformed a martingale X via a gambling strategy H into a new martingale H ·X. be real random variables. However, by Corollary 14.27, i the following map is measurable, $n-1 x \rightarrow Px [A] = \delta x \otimes \kappa ti$, $ti+1 \times n$ Bi. Hence * *) $E[\phi(X) A] = E[\phi(X) A] =$ X is exchangeable. At the first stage, we manipulate a coin at random such that the probability of a success (i.e., "head") is X. Use a contour argument similarly as in Theorem 2.45 to show that pc ≤ 34. 4 160 6 Convergence Theorems 6.3 Exchanging Integral and Differentiability of a success (i.e., "head") is X. Use a contour argument similarly as in Theorem 2.45 to show that pc ≤ 34. functions of two variables behave under integration with respect to one of the variables. For $\omega \in B$, define f (ω) = 0. n $\rightarrow \infty$ If τ is ergodic, then E[X0 | I] = E[X0]. Fix N \in N and choose N + 1 points of continuity y0 < y1 < . However, we want the medium not to have a homogeneous structure. martingales are submartingales. (ii) X is a continuous centered Gaussian process with $Cov[Xs, Xt] = s \land t$ for all s, $t \ge 0$. By assumption, we have $|f(t) - f(s)| \le C(ti) |t - s|\gamma \le C |t - s|\gamma$. Then there exists a kernel κ from (E, B(E)) to (E I, B(E)) to (E I, B(E)) to (E I, B(E)) to (E I) = s \land t for all s, $t \ge 0$. By assumption, we have $|f(t) - f(s)| \le C(ti) |t - s|\gamma \le C |t - s|\gamma$. Then there exists a kernel κ from (E, B(E)) to (E I) such that, for all $x \in E$ and for any choice of finitely many numbers 0 = j0 < j1 < j2 < . (iv) Let X and Y be supermartingales., Yt) and Xt := Ys . Show that, for any $\delta > 0$, the following two estimates hold:) * P Sn $\geq (1 + \delta)m \leq \exp - 0$. Proof We check properties (i)-(iii) of an algebra from Theorem 1.7. (i) $\Omega \in M(\mu *)$ is evident. The opposite is true for open sets: $\lim n \to \infty \delta 1/n$ ((0, ∞)) = 1 > 0 = $\delta 0$ ((0, ∞)). Remark 1.66 (iii) implies that (i) and (ii) also hold for A $\in M(\mu *)$ (with $\mu *$ instead of μ). Hence, the claim follows by Lemma 1.42. Hence (X, Y) has the density (see Example 1.105(ix)) 0 $\sigma 2$; ; y - $\mu 1 2\sigma 1 \sigma 2$ 2 2 -1/2 $\sigma 12$ (y - (x - $\mu 2$))2 + $\sigma 22$ (y μ 1) 2 = 4 π of σ 2 exp - 2 σ 12 σ 22 = Cx exp - (y - μ x) 2/2 σ x2. Definition 4.4 (Integral) If f: $\Omega \rightarrow [0, \infty]$ is measurable, then we define the integral of f with respect to μ by f d μ := sup I (g) : g \in E+ , g \leq f. If $\nu(\Omega) < \infty$, then the converse also holds. (18.15) The solution is (check this!) χ N (x) = (-1)N-1 (σ /2)N-1 (1 - x)2 UN-1 x/\sigma, (18.16) where define the integral of f with respect to μ by f d μ := sup I (g) : g \in E+ , g \leq f. If $\nu(\Omega) < \infty$, then the converse also holds. (18.15) The solution is (check this!) χ N (x) = (-1)N-1 (σ /2)N-1 (1 - x)2 UN-1 x/\sigma, (18.16) where define the integral of f with respect to μ by f d μ := sup I (g) : g \in E+ , g \leq f. If $\nu(\Omega) < \infty$, then the converse also holds. (18.15) The solution is (check this!) χ N (x) = (-1)N-1 (σ /2)N-1 (1 - x)2 UN-1 x/\sigma, (18.16) where σ = 0.5 m G/2 is measurable, then we define the integral of f with respect to μ by f d μ := sup I (g) : g \in E+ , g \leq f. If $\nu(\Omega) < \infty$, then the converse also holds. (18.16) where σ = 0.5 m G/2 is measurable. m/2! k m-k Um (x) := (-1) (2x)m-2k k =0 denotes the so-called mth Chebyshev polynomial of the second kind. Denote by M1 (E) the set of probability measures on E equipped with the topology of weak convergence (see Definition 13.12 and Remark 13.14). For topological spaces, these are the continuous maps, and for measurable spaces, these are the measurable maps. Denote by Z the set of finite partitions of Ω into pairwise disjoint measurable sets. (8.12) By Theorem 8.14(viii) (dominated convergence), there are null sets Br \in F, r \in Q, and C \in F such that 1 (8.13) lim F r + , $\omega =$ F (r, ω) for all $\omega \in \Omega \setminus Br$ n $\rightarrow \infty$ n as well as inf F ($-n, \omega$) = 0 Let N := sup F (n, ω) = 1 and n $\in N$ for all $\omega \in \Omega \setminus Br$ n $\rightarrow \infty$ n as well as inf F ($-n, \omega$) = 0 Let N := sup F (n, ω) = 1 and n $\in N$ for all $\omega \in \Omega$ C. \Rightarrow 2.2 Independent Random Variables. sup f \in F A Proof " \Rightarrow "Let F be uniformly integrable. Here also, X is a bounded martingale and we can compute the square variation process, \Rightarrow n-1 n *) 2 2) X*n = E (Xi - Xi - 1) Xi - 1 = 2 Xi (1 - Xi) A = 2 Xi (1 - X Also let $f' d\mu = f d\mu$ if this expression is defined for f. (iv) Show that XY $\delta \Rightarrow X$ for $\delta \downarrow 0$. Use an orthonormal basis b0,1, (cn,k), (dn,k) of suitably modified Haar functions (such that the cn,k have support [7, 1]) to show that a regular conditional distribution of WT given W1 is defined by P[WT $\in \cdot |W1 = x] = NT x_i T$. As $\varepsilon > 0$ was arbitrary, the integrals coincide. Lemma 21.5 Let X and Y be modifications of each other. Each letter is finished by a pause sign. In fact, An = ni=1 E[Yi2 Y1, . 1 (f + g + |f - g|) 2 and f Ag = 1 (f + g - |f - g|) 2 15.1 Separating Classes of Functions 329 Step 3. Theorem 15.55 Let $\mu \in \mathbb{R}$ and let C be a real positive definite symmetric d × d matrix. By virtue of the diagonal sequence (μk) k \in N and large k, we have [$\omega 1$, \in A such that $\alpha = \infty$ Ωn and such
that $\mu(\Omega n) < \infty$ for all $n \in N$. See [33] for a detailed treatment of finite exchangeable families. For any fixed $\omega 1 \in \Omega 1$, by the monotone convergence theorem, If ($\omega 1$) = limn $\rightarrow \infty$ Ifn ($\omega 1$) = limn x, then, in addition, (i) and (ii) are equivalent to: (iii) (|fn |p)n en is uniformly integrable and there exists a measurable f with meas fn - + f. Intuitively, such a small local change should not make a difference for a global phenomenon such as recurrence. For example, continuity is not among those properties. In particular, Student's t-distribution with $k \in N$ degrees of freedom is infinitely divisible (this is the case where $\sigma 2 = 1$ and $\theta = r = k/2$). Proof Assume that ($\mu \epsilon$) $\epsilon > 0$ with inf I B2r(x)(x) $\geq I(x) - 1$ $\delta \sup \varphi B2r(x)(x) \le \varphi(x) + \delta$. n=0 (iii) In this special case, fN (z) = $e\lambda(z-1)$ for $z \in C$ with $|z| \le 1$. X is called centered if E[Xt] = 0 for every $t \in I$. 19.17 Random walk on a hypercube. We call E[X] the intensity measure of X. 8 Show that if the walk starts at x, then the probability of visiting 1 before 0 is 17 using (i) the method of network reduction, and (ii) the method of matrix inversion. Takeaways The integral was defined first for functions which take only finitely many values. However, by (v), $E[Xn \ F] - E[X \ F] \le E[Zn \ F] \le E[$ a process has the (possibly time-inhomogeneous) Markov property if and only if past and future are independent given the present. 19.8. x 0 Fig. • Lemma 14.11 If every Ei is a n-system, then Z E , R is a n-system, then Z E , R is a n-system. sense that, for all $\varepsilon > 0$, lim max P[[Xn,1] > ε] = 0 and P[A *] = 0 and P[A *] = 1 could show up. Sometimes the ramified shape, in particular of the running martingale (French la martingale à martingale) and P[A *] = 0 and P[A *] Xk. Furthermore, there exists a $\delta > 0$ such that, for $x \in [-N, N]d$ and $u \in Rd$ with $|u| < \delta$, we have $1 - ei)u_x < \epsilon 2/6$. Using de Moivre's formula, one can show that, for $x \in (-\sigma, \sigma)$, $\chi N(x) = (-1)N-1 = (1 - x)2(\sigma/2)N-1$ sin N arc cos $x/\sigma 2(1 - x) 1 - (x/\sigma)2N-1$ sin N a -distributed. , n, and define stochastic kernels from Rd to Rd by κk (x, ·) = δx * μk for k = 1, . At the second stage, we toss the coin n times independently with outcomes Y1 , . ◆ Takeaways In the context of statistical mechanics, substantial contributions to an observable are not only due to the most frequent observations but also due to rare but very large observations. Lemma 4.3 The map I is positive linear and monotone increasing: Let f, $g \in E+$ and $\alpha \ge 0$. (iii) If $f \le g$, then I (f) $\le I$ (g). Example 1.74 The restriction of the Lebesgue-Borel measure λ on R, B(R)). More general criteria will be presented in Chap. We say D $n \rightarrow \infty$ that (Xn) $n \in N$ converges in distribution to X, formally Xn $- \rightarrow X$ or Xn \Rightarrow X, if the distributions converge weakly and hence if PX = w-lim PXn . n $\rightarrow \infty$ n Similarly, this also holds for x \in U \cap ($-\infty$, 0) with I (x) < ∞ ; hence lim inf n $\rightarrow \infty$ 1 log Pn (U) \geq - inf I (U). Infer that $\phi(t + s) = \phi(s)$ for all s \in Rd . Theorem 4.21 (Fatou's lemma) Let f \in L1 (μ) and let f1 , f2 , . . , D be independent Poisson processes with rate 1/D. This property of λ is called outer regularity. Definition 13.8 Let (E, dE) and (F, dF) be metric spaces. Using (15.5), we infer) * 2n $|u(2n-1)(\theta t)| \le 0$ almost surely. Assume that there exists an $\varepsilon > 0$ and an $C(\varepsilon) < \infty$ such that, for all s, $t \in I$ with $|t - s| \le \varepsilon$, we have $|f(t) - f(s)| \le 0$ $C(\varepsilon) | t - s| \gamma$. Lower bound For any
 $x \in E$ and r > 0, we have $e\varphi(\varepsilon d\mu\varepsilon \ge \lim \inf \varepsilon \log \varepsilon \rightarrow 0 \ e\varphi(\varepsilon d\mu\varepsilon Br(x)) - I(x) - \varphi(x) - I(x)$. (7.13) In order to show that (7.13) holds for all $g \in Lp(\mu)$, we first show $f \in Lq(\mu)$. Let a > 0 and $\tau = \inf t \ge 0$. Bt $\in \{0, a\}$., $Xn \sim \gamma p$ be independent geometrically distributed random variables with parameter $p \in (0, 1)$. Unlike the Poisson distribution, the normal distribution is the limit of rescaled sums of i.i.d. random variables (central limit theorem). (21.38) n=1 Compare (21.21). Thus $\epsilon > P[A \setminus F] = P[A \cap (\Omega \setminus F)] = P[A \cap (\Omega \cap F)]$ Let $(Xm,n)(m,n)\in N2$ be an independent family of Bernoulli random variables with parameter $p \in (0, 1)$. As a consequence we get a formal proof for the intuitive fact that increasing the resistance between any two given points. , xn)) = $P \times n = 1$, $(-\infty, xi] = n n$ $\mu i (-\infty, xi] = r \{i\}$ (xi). 12.3, we need some more technical tools (e.g., the notion of conditional independence). Let T > 0 be the temperature of the system and let $\beta := 1/T$ be the so-called inverse temperature. Proof First consider d = 1 and p < 1. By the martingale convergence theorem, X converges P-almost surely and in L1 (P)to a random variable X ∞ . (25.4) By Fubini's theorem and the dominated convergence theorem, we thus conclude that + E T 0, (Hs - Hsn)2 ds = Hs (ω) - Hsn (ω) 2 n $\rightarrow \infty$ (P $\otimes \lambda$)(d(ω , s)) - $\rightarrow 0$. For N \in N, define the truncated random variables |X| \wedge N. (iii) Let X and Y be independent Poisson random variables with parameters $\lambda \ge 0$ and $\mu \ge 0$, respectively. 19.2); that is, for all x, $y \in E$, we have $\pi(x) p(x, y) = \pi(y) p(y, x)$. Let X : $\Omega \to \Omega$ be a map. In the last section, we describe the speed of convergence to the equilibrium by means of the equilibrium by means of the equilibrium by means of the equilibr $\infty - \infty$ (2 π)-1/2 ey e-y 2/2 (15.1) n2 dy 2/2 = en . 493 493 497 500 502 506 510 xiv Contents 21 Brownian Motion . Then e-s ∞ 0 e-t ks kt g dt = ∞ e-t kt g dt \leq h. Let (bn)n \in N be an orthonormal basis of L2 ([0, 1]) such that Wt := limn $\rightarrow \infty$ nk=1 ξ k)1[0,t] , bk*, t \in [0, 1], is a Brownian motion. We come now to a theorem that combines Theorem 1.55 with the idea of Lebesque-Stieltjes measures. Formally, we argue that $\{X \in K\} \in Fs \subset Ft$ for all $s \leq t$. Proof Although the statement is intuitively so clear that it might not need a proof, we give a formal proof in order to introduce a technique called coupling. Then ∞ D X2n = Sn := Y1 + . Exercise 5.2.1 (Bernstein-Chernov bound) Let $n \in N$ and p1, $x \in I \circ (8.9)$ $x \in V$ Corollary 8.21 Let $p \in [1, \infty]$ and let $F \subset A$ be a sub- σ -algebra. The central limit theorem will show that the error is indeed exactly of this order. Hence, for all $t \ge 0$, $e-t x \nu^{\tilde{n}} n (dx) = un (t + 1) - u(t) . If \kappa 1 and \kappa 2 are (sub) stochastic, then \kappa 1 \otimes \kappa 2$ is (sub)stochastic. For $\delta > 0$ let $B\delta := x \in E : d(x, B) < \delta 13.1$ A Topology Primer 277 be the open δ -neighborhood of B. + Xn, kn \Rightarrow S, then S is infinitely divisible. \blacklozenge 2.2 Independent Random Variables of Definition 2.29 (Convolution) Let μ and ν be probability measures on (Z, 2Z). Define the stochastic matrix p (x, y) = ∞ n (x, y). Hence f is bounded, be real random variables with distribution functions F, F1, F2, .* Let An $\subset \Omega$ for any $n \in N$ and let $A \subset \infty$ n=1 An. Then A := {A $\subset \Omega$: A or Ac is countable} is a σ -algebra. This modification is called Brownian motion.* monotone limita, bU) a, bthe) * = limn $\rightarrow \infty$ E[Un] < ∞ . If X is irreducible and recurrent, then π and hence C are unique up to a factor. For $n \in N$, define gn = |fn - f|p. i=1 i=1 Since En $\uparrow \Omega$ and since μ and ν are lower semicontinuous, we infer $\mu(A \cap En) = \lim \nu(A \cap En) = \lim \nu($ (i) If ∞ n=1 P[An] < ∞ , then P[A] = 0. n $\rightarrow \infty$ (ii) If E[log(0)] < 0, then Xn $\rightarrow \infty$ a.s. $n \rightarrow \infty$ (iii) If E[log(0)] > 0, then Xn $\rightarrow \infty$ a.s. $n \rightarrow \infty$ (iii) If E[log(0)] > 0, then Xn $\rightarrow \infty$ a.s. $n \rightarrow \infty$ (iv) A constant of the second limits C that do not appear in the sum. One could conjecture that any point on the great circle is equally likely. $y \in E$ 19.1 Harmonic Functions 463 2. Consider a Galton-Watson process (Zn) $n \in N0$ with geometric offspring distribution p(k) = 2-k-1 for $k \in N0$. In order to get more precise estimates 3 for the integral, we need additional information; for example, the value V1 := f 2 (x) dx - I 2 if f \in L2 ([0, 1]). The case of a right boundary point is similar. Theorem 14.8 Let I be countable, and for every i \in I, let (Ω i, τ i) be Polish with Borel σ -algebra Bi = σ (τ i). (iii) (Triangle distribution) Note that Tria = U[-a/2,a/2] * U[-a/2,a/2]; hence ϕ Tria (t) = ϕ U[-a/2,a/2] (t) = 4 sin(at/2)2 1 - cos(at) = 2. n}: Sk \geq t and Ak = { $\tau = k$ } for k = 1, . Let z = u - iv be the complex conjugate of z and |z| = u2 + v 2 its modulus. The Metropolis algorithm constitutes a universal tool for the construction of such a Markov chain. For each n \in N, let (Xn,k) k \in N be an independent family of random variables with Xn,k \sim Berpn,k . 9.2 Martingales Everyone who does not own a casino would agree without hesitation that the successive payment of gains Y1, Y2, C(x) C(x) V(x) = C(x, y)/C(x). D Show that if XZ = Y Z holds, then X = Y . Hence, by the preceding theorem, for any $n \in N$, there exists an open setUn $\supset A \cap Kn$ with $\lambda(Un \setminus A) < \varepsilon/2n$. Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) *) * Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) *) * Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) *) * Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) *) * Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) *) * Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) *) * Definition 9.15 (Stopping time) A random variable τ with values in I \cup { ∞ } is called a stopping time (with respect to F) if for any t \in I { $\tau \leq t$ } \in Ft. Therefore,) * Definition 9.15 (Stopping time) A random variable τ (Stopping time) A random varia could follow the strategy H -H (which gives a final payment of VT -VT = 0) and make a sure profit of v0 - v0. The distribution. (See Figs. Corollary 15.10 A finite measure μ on Zd is uniquely determined by the values $\phi\mu$ (t) = ei)t, x* μ (dx), t \in [$-\pi$, π)d. (iv) As ei)t, X* and ei)t, Y* are independent random variables, we have *) *) *) $\phi
X + Y(t) = E ei)t, X^* = ei)t, X^* = E ei)t, X^* = ei$ t, X^* = ei)t, X Markov Property .. If I is an interval, then a map $g: I \rightarrow R$ is called affine linear if there are numbers $a, b \in R$ such that g(x) = ax + b for all $x \in I + Xn$ for $n \in N$. Such a risk-free profit (or free lunch in economic jargon) is called an arbitrage. probability theory, Jensen's inequality for convex functions, and indicate how to derive from it Hölder's inequality and Minkowski's inequality. Then (Y n)n \in N is an indepen-n dent family of random variables (with values in C([0, 2 m])). Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into Universitext. 266 12 Backwards Martingales and Exchangeability Takeaways A backwards martingale (Yn)n \in N0 is a martingale (Yn)n \in N0 is a stochastic process such that (Y-m)m \in -N0 is a martingale (Yn)n \in N write ZI , (14.3) Z= J \subset I finite and similarly define Z R and Z E all f, g \in C, f vg = are also in C. x \rightarrow P Xj \leq xj for all j \in J = P j \in J Then FJ is called the joint distribution function of (Xj)j \in J. (23.18) x \in Remark 23.18 (Moment condition) The tail condition (23.17) holds if there exists an $\alpha > 1$ such that lim sup ε log $e\alpha \phi/\varepsilon$ du $\varepsilon < \infty$. (23.19) $\varepsilon \rightarrow 0$ Indeed, for every M \in R, we have $e\phi(x)/\varepsilon 1{\phi(x) \geq M}$ using (dx) = M + ε $\log \epsilon \log e(\varphi(x) - M)/\epsilon 1 \{\varphi(x) \ge M\} \mu \epsilon (dx) \le M + \epsilon \log e\alpha(\varphi(x) - M)/\epsilon \mu \epsilon (dx) = -(\alpha - 1)M + \epsilon \log e\alpha(\varphi(x)/\epsilon \mu \epsilon (dx))$. Takeaways We recognised the function spaces L2 as Hilbert spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined of the function spaces. In fact, it is not hard to show that M is the expected determined determined of the function space. In fact, it is not hard to show that M is the expected determined dete time to return to 0; hence the criterion for positive recurrence could also be deduced by Theorem 17.52. For any $y \in A$ and $x \in (BR(y)/3(y)) \cap Qn$, we have $R(x) \ge R(y) - x - y^2 > 213 R(y)$, and hence r(x) > 3 R(y) and thus $y \in Br(x)$ (x). By (21.17) and (21.18), using the convergence theorem for backwards martingales (Theorem 12.14), we get that in the sense of L1 -limits) * EBT [F (B)] = lim Ex F (BT n + t)t ≥ 0 FT n n $\rightarrow \infty$) *) * = lim Ex F (BT + t)t ≥ 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t ≥ 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 FT n = Ex F (BT + t)t = 0 is true by definition. (ii) $E[X | F] \in L1$ (P), and for any $A \in F$, we have E[X | F] dP = A X dP.; ; A 1 and [0,n] ∞ d(f, g) = ∞ 2-n dn (f, g)., Yk), where, given $\Xi \infty$, the random variables Y1 , . We have shown the existence of solutions of the Dirichlet problem in Example 19.4. In order to show uniqueness (under certain conditions) we first derive the maximum principle for harmonic functions. At any time $n \in N0$, one site In out of Λ is chosen at random and the individual at that site reconsiders his or her opinion. d $\lambda k \lambda = 0$ (21.40) 21.9 Pathwise Convergence of Branching Processes 555 In particular, the first six moments are Ei [Zn] = i, Ei [Zn 2] = 2i n + i 2, Ei [Zn 3] = 6i n 2 + 6i 2 n + i 3, Ei [Zn 4] = 0 [Zn4] = 24i n3 + 36i 2 n2 + (12i 3 + 2i) n + i 4, (21.41) Ei [Zn5] = 120i n4 + 240i 2 n3 + (120i 3 + 30i) n2 + (20i 4 + 10i 2) n + i 6. Therefore, for every k = 2, 19.11 Steps 3 and 4. Define Sn := Z1 + . The claim follows since, for $y \in Zd$, $[-\pi,\pi)d \in i)t, y-x^* dt = (2\pi)d$, 0, if x = y, else. Then E[Xn λ] = 0 and Var[Xn λ] = 0, C λ^* . Hence $A \in G$. It is implicitly assumed that the reader has a certain familiarity with the basic concepts of probability theory, although the formal framework will be fully developed in this book. (18.9) See Fig. Then h1, hn-1 \in UL are distinct but connected in K p even if we remove x. Show that this random walk in a random environment is • a.s. transient if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log(0)] = 0$, e.s. null recurrent if $E[\log(0)] = 0$, and • a.s. positive recurrent if $E[\log$ B. In the case of finite signed measures this $n \rightarrow \infty$ is equivalent to: (µ) is bounded and µn (A) $- \rightarrow \mu(A)$ for any measurable A (see [38, Theorem IV.9.5]). μ is infinitely divisible if and only if there exists an $\alpha \ge 0$ and a σ -finite measure 16.1 Lévy-Khinchin Formula 373 $\nu \in M((0, \infty))$ with $(1 \land x) \nu(dx) < \infty$ (16.5) and such that $u(t) = \alpha t + 1 - e - t x \nu(dx)$ for $t \ge 0$. Let and P = A = A P. Here R1 = 1910, R2 = 25, R3 = 1, $\delta = 125$, $R1 = \delta/R1 = 25$, $3 = \delta/R3 = 513$. Theorem 7.10 Let $G \subset Rn$ be open and convex and let $\phi: G \to R$ be a map. 6.1 Almost Sure and Measure Convergence 149 Then (since $d(f, g) > \epsilon$ } $n \to \infty \le \mu$ Am $\cap \{d(f, g) > \epsilon\}$ $n \to \infty \le \mu$ Am $\cap
\{d(f, g) > \epsilon\}$ $n \to \infty \le \mu$ Am $\cap \{d(f, g) > \epsilon\}$ $n \to \infty \to \mu$ Am $\cap \{d(f, g) > \epsilon\}$ $n \to \infty \to \mu$ Am $\cap \{d(f, g) > \epsilon\}$ $n \to \infty \to \mu$ Am $\cap \{d(f, g) > \epsilon\}$ $n \to \infty \to \mu$ Am $\cap \{d(f, g) > \epsilon\}$ $n \to \infty \to \mu$ Am $\cap \{d(f, g) >$ $\epsilon/2$ + μ Am \cap {d(g, fn) > $\epsilon/2$ } - $\rightarrow 0$. **E**[X] = R Exercise 5.1.3 Let X ~ β r,s be a Beta-distributed random variable with μ (E \ L) ≤ 1 for all $n \in \mathbb{N}$. Theorem 5.7 The map Cov : L2 (P) × L2 (P) $\rightarrow \mathbb{R}$ is a positive semidefinite symmetric bilinear form and Cov[X, Y] = 0 if Y is almost surely constant. Thus, if the defining equality (or inequality) holds for any time step of size one, by induction it holds for all times. Assume that in the beginning there is one black and one red ball in the urn. & Exercise 21.2.4 Let B be a Brownian motion, a < 0 < b. & 6.2 Uniform Integrability From the preceding section, we can conclude that convergence in measure plus existence of L1 limit points implies L1 -convergence. Upper bound For M > 0 and $\epsilon > 0$, define $\epsilon \varphi(x)/\epsilon \epsilon e \mu \epsilon (dx) = FM \vee GM$. Show that X1 \in L1 (P) and Y = E[X1] almost surely. Hence k=1 E $\cap A \subset \infty$ n=1 Bn, E $\cap A \subset C$ mn ∞ n=1 k=1 Cnk and En = Bn mn k=1 Cnk. Hence, it is enough to show that, for Brownian motion X, we have Cov[Xs, Xt] = min(s, t). Proof (i) This is obvious. Show the conditional Borel-Cantelli lemma: P[A A +] = 0. \in (1, 2], let dn = n E[X] for all $n \in N$. If μ is a probability measure, the existence of the sequence (Ωn) $n \in N$ is not needed. 150 6 Convergence Theorems \tilde{n} fn $n \to \infty$ " = "Assume d(f, $- \to 0$. $\delta \omega$ is called the Dirac measure for the point ω . Similarly, we get Var[X] = E[X2] - $\mu 2$ = . Thus, for N \in N and $0 \leq t1$, . They can be generated by classes with a different structure (e.g., a topology). Theorem 17.59 (Strassen's theorem) Let L := (x1, x2) \in Rd × Rd : x1 ≤ x2. Lemma 21.45 The moments of Zn are Ei [Znk] = (-1)k d k (n) $\rightarrow \lambda$ i ψ (e) . n $\rightarrow \infty$ "(iii) \Rightarrow (i)" Since |fn|p $\rightarrow \rightarrow$ |f |p in measure, by Theorem 6.25, we have |f |p \in L1 (μ) and hence f \in Lp (μ). be independent, square integrable random variables with E[Yn] = 1 for all n \in N. x Note that h(x)/(1 \land x) ≤ 1 for all x > 0. (3.5) (iv) Let X1 , . The family (κt (x, A), $t \in I$, $x \in E$, $A \in B(E)$) is also called the family of transition probabilities of X. Hence, also in this case, i is continuous. A is a semiring and σ (A) = B(R), where B(R) is the Borel σ -algebra on R. In particular, this implies that not all exponential moments are finite. Clearly, $\nu = \nu a + 3 \nu s$ and νs ($\Omega \setminus E$) = 0; hence $\nu s \perp 3\mu$. By $Q \pm (A) :=$ $E[X \pm 1A]$ for all $A \in F$, we define two finite measures on (Ω, F) . $n \rightarrow \infty$ Proof "(i) $= \{Sn \rightarrow \infty\}$ is an invariant event and thus has probability either 0 or 1. By construction, equation (13.3) holds and the proof is complete. D Note that F1n (x) := F1 \circ . Let $Ei \in \sigma$ (Ei) for any $i \in J \setminus (J \cup \{j\})$. 21.8 Donsker's That is, we do not have control on the quantity $P[|I:n - I| > \varepsilon]$. Fix a parameter $\rho \in [0, 1]$ and independently declare any edge of L2 open with r 1 - p. Recall that this is a probability measure on the space of sequences (E N0, B(E) \otimes N0), (iii) There is a map $h \in L1$ (u), $h \ge 0$, such that |f| (\cdot a.e. for all $x \in I$. Let $G \subset F \subset A$ be σ -algebras and let $Y \in L1$ (Ω, A, P). Hence we are looking for solutions VT - 1 + HT (XT - XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT -, VT - 1 + HT (XT - 1 + XT - 1) = VT - 1 + YT - 1 (VT - 1 + YT - 1) = VT - 1 + YT - 1 (VT - 1 + YT - 1) = VT - 1 + YT - 1 (VTall $n \in N$ and all $\in S(n)$, $E[\phi(X)|En] = E[\phi(X)|En]$.) and (A2, A4, A6, ., n defines a substochastic kernel ii// $\kappa k := \kappa 1 \otimes \cdots \otimes \kappa i$ from ($\Omega 0$, A0) to $\times \Omega k$, Ak. In this section, we study a criterion for relative sequential compactness in L1, the so-called uniform integrability. However, then also Xn = Xn0 for all $n \ge n0$., Ym) by Yi := #{k = 1, . By Jensen's inequality, we get $Var[X] = E[X2] - (E[X])2 \ge 0$. (iv) (N.N.) This can either be computed directly or can be deduced from (iii) by using the Fourier inversion formula (equation (15.2)). For $f \in Cc$ (E), there exists an $0 \in N$ such that the support of f is contained in Wn0. Theorem 20.35 (Kolmogorov-Sinai) Let P be a generator of A; that is A = -n (P). Definition 18.14 Define a stochastic matrix p on E by $p(x, y) = \int \left\{ q(x, y) \min 1, \left| 1 - \pi(y)q(y,x) \pi(x)q(x,y) + 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) > 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) > 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) > 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) > 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) > 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) > 0, 0, p(x, z), z=x \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0, \text{ if } x = y, q(x, y) = 0,
\text{ if } x = y, q(x, y) = 0, \text{ if } x = y,$ locally compact Abelian group. Then (Xi, j) $\in I \times J$) is uniformly integrable. (iii) If (Ei $\cup \{\emptyset\}$) is \cap -stable, then (Ei)i $\in I$ is independent. Show that AX ~ NAµ,ACAT for every $m \in N$ and every real $m \times d$ matrix A. - := PX is called the geometric distribution2 with parameter Then $\gamma p := b1, p$; formally $\gamma p = \infty p(1 - C)$ p)n δn . TV n Indeed, the probability pn,k that we do not see any ball twice when drawing k balls (with replacement) from n different balls is pn,k = k-1 (1 - l/n) l=1 and thus Rn,k $\geq 2(1-pn,k)$. (vii) For any $A \in F$ and $B \in A$, we have $P[A \cap B] = 0$ if P[A] = 1. A b-adic prefix code is defined in a similar way as a binary of the second secon prefix code; however, instead of 0 and 1, now all numbers 0, 1, . Let $Bn = \{d(fn, fn+1) > \epsilon n \}$ and $B = \lim sup Bn \cdot N \rightarrow \infty$ Hence $XY \in L1$ (P). 2 The entropy of the state m is H (m) = -1 + m 1 - m 1 general situation. $= [-n,n] n \rightarrow \infty$ Exercise 13.2.5 Let E = R and $\mu n = \delta n$ for $n \in N$. Let NL1 be the number of infinite open clusters if p we consider all edges e in EL as open (independently of the value of Xe). We say that two points $x, y \in Zd$ are connected by an open path if there is an $n \in N$ and an open path ($x0, x1, \ldots Example 1.107$ (i) Let $\theta, r > 0$ and let $\Gamma \theta$, r be the distribution on $[0, \infty)$ with density $x \to \theta r x r - 1 e - \theta x$. .). Exercise 20.6.1 Let $\Omega = [0, 1)$ and $\tau : x \to 2x$ (mod 1). Takeaways A Feller semigroup of stochastic kernels is a Markov semigroup of stochastic kernels is a Markov semigroup with just enough additional regularity such that we can construct an RCLL version of the corresponding Markov process. $j \in J \in J$ Reflection How do you choose four events A1, A2, A3, A4 such that each pair Ai, Aj, i = j, and each triple Ai, Aj, k = 3, is independent, but A1, A2, A3, A4 is not? For $\tau - s$, since τ is a stopping time, we have $\{\tau - s \le t\} = \{\tau \le t + s\} \in Ft + s$. + Rn-1)-1 (see Fig. By (8.12) and (8.13), we have $F^{(\tau)}(z, \omega) = F(z, \omega)$ for all $z \in Q$ and $\omega \in \Omega \setminus N$. Exercise 21.2.5 Let B be a Brownian motion, b > 0 and $\tau b = \inf\{t \ge 0 : Bt = b\}$. Consider the set of functions 1 0 $g d\mu \le v(A)$ for all $A \in A$, $G := g : \Omega \rightarrow [0, \infty] : g$ is measurable and A 7.5 Supplement: Signed Measures 185 and define $\gamma := \sup g d\mu : g \in G$. For $n \in N$, the nth derivative ψX fulfills lim $\psi X(n)(z) = z \uparrow 1 \infty$ $P[X = k] \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot k(k - 1) \cdot \cdot (k - n + 1) \cdot (k - n$ 1), (3.2) k=n where both sides can equal ∞ . 9.3 Discrete Stochastic Integral. Further, let Y1 = 2 if X1 = 1 and Y1 = -1 otherwise. "(vi) \Rightarrow (iii) ν is totally continuous with respect to μ . Define Y = (Y1, . We will derive a large deviations principle for the empirical measures 1 δ Xi. Exercise 1.3.3 Let (µn) $n \in N$ be a sequence of finite measures on the measures on the measures on the measures. However, if we do not intersect with the set A, then stochastic convergence would fail, although a.e. we still had fn \rightarrow f. Since κ^2 is finite, we have $n \ge 1 \, A\omega 0$, $n = \Omega 1$ for all $\omega 0 \in \Omega 0$. In order to check the assumptions of Theorem 1.53, we only have to check that μ is σ -subadditive. \blacklozenge Theorem 17.25 Let q be an E × E matrix such that $q(x, y) \ge 0$ for all x, $y \in E$ with x = y. be exchangeable real random variables. If there exists a random variables is upper semicontinuous and finite (Theorem 1.36). We assume that {L, X1, X2, . With different bounds (instead of 0.8), the statement was found independently by Berry [10] and Esseen [46]. Reflection Find an example of a discontinuous linear map $F: V \rightarrow R$. In particular, $A* := \lim \text{ supn} \rightarrow \infty \text{ An} \ A = E \phi k - 1$ (X1, . To this end for $n \in N0$, define Mn := X0 + n Xk - E[Xk Fk-1] (10.1) k=1 and An := n E[Xk Fk-1] (10.1) k=1 and An := n E[Xk Fk-1] - Xk-1. Takeaways For two examples of aperiodic and irreducible Markov chains, we have constructed a coupling such that two chains meet almost surely: Random walks on the d-dimensional integer lattice and positive recurrent Markov chains. Corollary 13.31 If E is a locally compact separable metric space, then $M \le 1$ (E) is varuely sequentially compact. Consequently, F (x) =)x, u*/u2. Proof Evidently, the singleton { $|X0|p}$ is uniformly integrable. Then there exists an $\varepsilon > 0$ with (x - ε , x + ε) \subset U. 234 10 Optional Sampling Theorems Proof It is enough to show that E[XT 1A] = E[XT 1A] for all A \in $F\tau$. be open sets and let $n \ C \subset \infty i=1$ Ai. Even if all moments exist, the distribution of X is, in general, not uniquely determined by its moments. In general, not uniquely determined by its moments. In general, not uniquely determined by its moments. In general, the expectation could be zero for some t. Exercise 21.6.1 Show that the map $F\infty : \Omega \to [0, \infty]$, is A-measurable. $\in M1$ (Rd) with characteristic functions ϕ , $\phi 1$, $\phi 2$, . Then, for I = (a, b], clearly PNI = $Poi\alpha(b-a) = Poi\alpha(I)$. For $t \in I$, define the σ -algebras that code the past before t and the future beginning with t by $F \leq t := \sigma$ (Xs : $s \in I$, $s \geq t$). Reff ($0 \leftrightarrow x$) + Reff ($0 \leftrightarrow x$ Obviously, only points of UL can be leaves. More precisely, $\Lambda * (x) = t * (x)$, $\Lambda * (x) > t$, $x^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) > t$, $x^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) > t$, $x^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) > t$, $x^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) > t$, $x^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) > t$, $X^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) > t$, $X^* - \Lambda(t)$ for all t = t * (x), $\Lambda *
(x) > t$, $X^* - \Lambda(t)$ for all t = t * (x), $\Lambda * (x) = t$, $\Lambda = 0$, Λ $\Omega \in A$, (ii) for any two sets A, $B \in A$ with $A \subset B$, the difference set $B \setminus A$ is in A, and (iii) ∞ n=1 An $\in A$ for any choice of countably many pairwise disjoint sets A1, A2, Lemma 7.46 Let μ , ν be finite measures on (Ω , A) that are not mutually singular; in short, $\mu \perp \nu$. Exercise 7.5.1 Let μ be a σ -finite measure on (Ω , A) and let ϕ be a signed measure on the sum of t (Ω, A) . The Lebesgue integral approximates the area by the measure of the levels sets (right hand side). A similar argument for $q - yields \lim inf Fnk(x) \ge k \rightarrow \infty F(x)$. To this end, we compute $\#A^{\sim} \#B \#B^{\sim} P[A] = \#A 36 = 6$ and P[B] = 36 = 6. Let $r(x) \in (R(x)/2, R(x)) \cap Q$. Uniqueness of the decomposition is easy: If x = y + z is an orthogonal decomposition, then $y - y \in W$ and $z - z \in W \perp$ as well as y - y + z - z = 0; hence 0 = y - y + z - z = 0; hence y = y and z = z. Hence $\mu \{d(f, g) > 0\} = 0$. More information about this series at Achim Klenke Probability Theory A Comprehensive Course Third Edition Achim Klenke Institute (d(f, g) > 0) = 0. More information about this series at Achim Klenke Probability Theory A Comprehensive Course Third Edition Achim Klenke Institute (d(f, g) > 0) = 0. f"ur Mathematik Johannes Gutenberg-Universit"at Mainz, Germany ISSN 0172-5939 ISS under exclusive license to Springer Nature Switzerland AG 2020 This work is subject to copyright. Let $x(i) \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the opinion of the voter at site $i \in \Lambda$ and denote by $x \in \{0, 1\}$ be the voter at $x \in \{0, 1\}$ be the voter at N. $\mu(A) \leq n=1$ n=1 We now give two different proofs for (1.13). Note that, for all $\epsilon > 0$, |z|, ..., kn it x $e - 1 \leq 2 \ge 2/\epsilon^2$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| > \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon$, $\epsilon \mid t|$, if $|x| < \epsilon \mid t|$, if $|x| < \epsilon$, if |x| < $= \alpha \cdot (ti - ti - 1)$ and show that (Nti - Nti-1) i=1,...,m is independent (5.18) and Nti - Nti-1 ~ Poi\lambdai for all i = 1, . Clearly, the probability qn := P[Zn = 0] that Z is extinct by time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n is monotone increasing in n. We let X0 = x0 > 0 and for n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. Theorem 13.25 (Continuous of the time n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. The time n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. The time n = 1, . In particular, dv the density dµ is unique up to equality µ-almost everywhere. The time n = 1, . In particular, dv the density qµ is unique up to equality µ-almost everywhere. The mapping theorem) Let (E1, d1) and (E2, d2) be metric spaces and let $\phi: E1 \rightarrow E2$ be measurable. n=1 Takeaways Kolmogorov's inequality gives bounds for the maximum of partial sums of random variables similar to Chebyshev's inequality for one random variable. Remark 21.34 Sometimes we want a Brownian motion to start not at X0 = 0 but = at an arbitrary point x. t \leq T Theorem 10.11 (Optional sampling theorem) Let X = (Xn) n \in N0 be a supermartingale and let $\sigma \leq \tau$ be stopping times. Let X be a symmetric simple random walk on Zd. (ii) Now let I \subset R be an interval and let X and Y be almost surely right continuous. We have established that every finite measure on R, B(R) is a Lebesgue-Stieltjes measure for some function F. For $k \in N$ let mk (Xn) = E[Xnk] be the kth moment if Mk (Xn) < ∞ . Then (Xn,l) is independent, centered and normed. Clearly, an is an n-symmetric for any m > n). Exercise 21.3.1 (Hard problem!) Let Px be the distribution of Brownian motion started at $x \in R$. Hence E[X] is where none of the traits is favored by selection. In Corollary 12.19, we will see that in the case of independent random variables, E is also P-trivial., Xn)-measurable. 3 gc/3 dµ 0. A graphical representation of the points ($F\Phi-1$ (t)), t \in R is called Q-Q-plot or quantile-plot. n $\rightarrow\infty$ 21.8 Donsker's Theorem 551 Next, for N > 0 and s, t \in [0, N], we compute the
fourth moments of the n, n⁻ Kn, n for the main term. Depending on the outcome of a random experiment, we choose the distribution of a second random experiment, we choose the distribution of a second random experiment, we choose the distribution of a second random experiment, we choose the distribution of a second random experiment, we choose the distribution of a second random experiment, we choose the distribution of a second random experiment (in a measurable way). In order to establish the existence of μ , we define as in Lemma 1.47 $\mu * (A) := \inf \mu(F) : F \in U(A)$ for any $A \in 2\Omega$. Later we will encounted more 0-1 laws (see, for example, Theorem 2.37)., in $\in \{1, . Hence, let (fnk) \in \mathbb{N} \ (i) \ (Uniform distribution) \ This is immediate. In addition, we also want to construct systematically infinite families of random variables with given (joint) distributions., Ak <math>\in \mathbb{A}$ with $-l\Omega = A1 \cup . 11.2$ for a simulation of the voter model. Show that μ is absolutely continuous with bounded continuous density $f = d\mu d\lambda$ given by 1 f (x) = $2\pi \propto -\infty e^{-itx} \phi \mu$ (t) dt for all $x \in R$. If, on the other hand, $A \subset R$ is not in B(R), then $A \in 2R$, but X-1 (A) $\in B(R)$.) (ii) For $x \in R$, we agree on the following notation for rounding: $x! := max\{k \in Z : k \leq x\}$ and " $x# := min\{k \in Z : k \geq x\}$. Let τ be the shift on Ω and let P be an invariant probability measure. 19. (17.19) Thus, simple random walk on Z is recurrent if and only if it is symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. n=1 (Xt, $t \in N0$) is called a symmetric; that is, if p = 12. 1], define $Xtn = 1[0,t](s)n \lambda(ds) = \xi m bm(s) m = 1 n \xi m)1[0,t]$, bm *. be independent real random variables. By Kirchhoff's rule, we have I(l, l + 1) = -I(x1) for any l = 0, . Exercise 17.7.1 Use an elementary direct coupling argument to show the claim of Theorem 17.61 for the case $n2/n1 \in N$. Exercise 15.5.1 The argument of Remark 15.39 is more direct than the argument with Lévy's continuity theorem but is less robust: Give a sequence X1, X2, There exists a unique smallest σ -algebra A* \supset A and an extension μ * of μ to A* such that (Ω , A*, μ *) is complete., 2n }, we have *) P Xk2-n - X(k-1)2-n $\geq 2-\gamma$ n $\leq C 2-n(1+\beta-\alpha\gamma)$. More generally, for $J \subset J \subset I$, the restricted map XJ $\times \Omega \rightarrow \times \Omega$, $j j \in J j \omega \rightarrow \omega j \in J J$ (14.1) is called the canonical projection. Evidently, $A \cap \tau - k$ (A) = A for every $k \in \mathbb{N}0$. "(ii) \Rightarrow (iii)" For any $\epsilon > 0$, there is an $n\epsilon \in \mathbb{N}$ such that $fn - fn\epsilon 1 < \epsilon$ for all $n \ge n\epsilon$. (7.8) 7 Lp -Spaces and the Radon-Nikodym Theorem 178 If in (7.7) we choose $h = 1 \{g1\}$ in (7.8), we obtain that $(\mu + \nu)$ -almost everywhere $g \le 1$. Theorem 16.27 Let $\mu \in M1$ (R) be nontrivial. (2) Checking if F is constant needs computer time of the same order of magnitude. (i) (Pi $\circ X0-1$, $i \in I$) is tight; that is, for every $\varepsilon > 0$, there is a K > 0 such that in order for)X* to have the simple form as in Example 10.6, it is not enough for the random variables Y1, Y2, Therefore, $X \propto$ is the Radon-Nikodym density of Q with respect to P. In the physical literature, $Tc := 1/\beta c$ is continuous at yi = xi. n=1 By Parseval's equation and the Bienaymé formula, we have $f 22 = \infty *)*)$)f, $bn * 2 = Var I (f) = E I (f) 2 \cdot 7.1$, these inequalities enabled us to show the celebrated Fischer-Riesz theorem. Let A := {device is defective}, B := {device is defective}, and P[B] = 0.02, P[A|B c] = 0.98, P[A|B c] = 0.95, P[B c] = 0.98, P[A|B c] = 0.95, P[B c] = 0.98, P[A|B c] = 0.95, P[B c] = 0.98, P[A|B c] = 0.98, P[A| some $\lambda > 0$ and then find the λ that optimizes the bound. (i) The canonical triple of X1 + . x (ii) Let $f: R \rightarrow [0, \infty)$ be continuous and let F(x) = f(t) dt for all $x \in R$. < bn and $A = i=1 \mu(A) = n$ (bi - ai). #En This implies the first inequality in (23.15). Let $\varepsilon > 0$. E is called locally compact if any point $x \in R$. < bn and $A = i=1 \mu(A) = n$ (bi - ai). #En This implies the first inequality in (23.15). compact. In Definition 2.32, we defined the convolution of two real probability measures μ and ν as the distributions μ and ν, respectively. In analogy with Lemma 8.10, we make the following definition. Definition 2.11 Let X and Y be stochastic processes on (Ω, A, P) with time set I and state space E. $n \rightarrow \infty$ Note that $kn+1 \leq (1 + 2\epsilon)kn$ for sufficiently large $n \in N$. be probability measures on E. & Exercise 13.1.5 Let $C \subset Rd$ be an open, bounded and convex set and assume that $U \subset \{x + rC : x \in Rd, r > 0\}$ is such that $W := U \in U \cup I$ has finite Lebesgue measure λd (W). For a fixed realization of the repeated experiment, let $\omega 1$, $\omega 2$. Now define Xn (k) := NI n (k) and 0 X n (k) := 1, 0, if Xn (k) \geq 1, else. (For bounded f, V1 can easily be bounded.) Indeed, in this case, Var[I:n] = V1 /n; hence, by Chebyshev's inequality, ('P |I:n - I| > ϵ n - 1/2 \leq V1 / ϵ 2., n n 1 - e- θ i x. Before we do so, we make the following definition. (14.8) k=1 Proof For k = 1, . f 1 \in 1 Show that H (p) \leq H (p1) + H(p2). (i) Show that there exist finite numbers $(dk)k\in N$ (depending on the distribution PX1) such that for any k, $n \in N$ we have $1 \times L(x_1 + 1) + H(p_2)$. (ii) A is closed under intersections. For monotone increasing bounded f: $3Rd \rightarrow R$, 3 3we have $f(x_1) - f(x_2) \le 0$ for every $x = (x_1, x_2) \in L$; hence $f d\mu 1 - f d\mu 2 = f(x) - f(x) \phi(dx) \le 0$ and thus $\mu 1 \le st \mu 2$. Denote $\phi(X) = \phi(X1, p \text{ Hence A is in the tail } \sigma \text{ -algebra } T((Xe) e \in E)$ by Theorem 2.35., $\omega n]c = [\omega 1, . 12.3, here we give a different proof of de Finetti's theorem 2.35.$ (Theorem 12.26). If $\{\omega\} \in A$, then the completion is $A^* = 2\Omega$, $\mu^* = \delta\omega$. Reflection If instead of $\mu(N0) = 1$, in the previous lemma we only assume $\mu(N0) \in [0, 1]$, then we still have (i) \Rightarrow (ii), \Rightarrow the shift on the product space $\otimes I$. The nth symmetrized average An (ϕ) : E N \rightarrow R, x $\rightarrow 1$ $\phi(x)$ n! (12.1) \in S(n) is an n-symmetric map. We still have to show that κ is a version of the conditional distribution. For x \in Z \ {0}, use partial integration to compute the integral, $\pi - \pi \cos(tx) \phi(t) dt = 2 \pi \cos(tx) (1 - 2t/\pi) dt 0 = \pi 4 2 4 4 1 - \sin(\pi x) - \sin(0) + \sin(tx) dt x \pi x \pi x 0 = 4 (1 - \cos(\pi x))$ Furthermore, tl \in Dl for l = n, . Here summation is over all subsets of {1, . In particular, for $\lambda \in (0, 1)$, R $\lambda := \infty$ $\lambda n \text{ pn}(0, 0) \text{ n}=0 = (2\pi) - D = (2$ $F\Phi-1$ (t) $-4 - 20 \alpha = 0.4824 *$ from Example 15.53 with $\alpha = 0.48$ (right). Since all Xi \circ Y are measurable, we have Y -1 (A) \in A for any A \in E . To conclude, we pick up again the example with which we started. Hence ye is minimal for $\varepsilon = \varepsilon 0$. In fact, this argument is even more robust since it uses only that the single steps of X have an expectation that is not zero. Theorem 4.17 The map \cdot 1 is a seminorm on L1 (μ); that is, for all f, $g \in L1$ (μ) and $\alpha \in R$, $\alpha f 1 = 0$ if f = 0 a.e. Proof The first and the third statements follow from Theorem 4.8(i). If the derivative of f is bounded, then f is also (globally) Hölder-1-continuous, but not necessarily (globally) Hölder- γ -continuous for any $\gamma \in (0, 1)$. 408 17 Markov Chains In particular, we see that Xt is finite for all t. Manifestly, the map $x \to Ex [F (B)] = E0 [f (Bt1 + x, . Furthermore, we have Px [Xt \in A] = (\delta x \cdot \kappa t)(A) = \kappa t (x, A)$. Let $\Omega = \Omega + \Omega - be$ a Hahn decomposition of Ω . Hence, we get kn 3 t2 ϕ n, l (t) -1 = -. Theorem 24.4 Let X be a random measure on E. Let A0 = { \emptyset }. Concluding, we get h(P, τr) = 0. Our aim is to find a maximal element f in G (i.e., an f for which This f will be the density of νa . In particular, any edge that directly connects 0 to 1 can be deleted. ai Then μf is a σ -finite content on A (even a premeasure). It is impossible that the random walk would go to ∞ (or $-\infty$) slower than linearly., Xn-1, Dn) for any n = 1, . Do the sets $F^n = [-n, n] \cap Z$, $n \in N$ do the trick? Denote by U ϕ the set of points of discontinuity of ϕ . Thus $X\lambda = \lambda$, X^* . be i.i.d. with E[Yi] = 0 and E[Yi2] = 1. "(i) \Rightarrow (ii)" Note that $|x + y|p \le 2p$ (|x|p + |y|p) for all $x, y \in R$. Show that A is a semiring and
μ is a content on A that is lower and upper semicontinuous but is not σ -additive. 19.9 Star-triangle transformation.) * n $\rightarrow \infty$ A coupling is called successful if P(x,y) $\rightarrow 0$ for all m $\geq n$ {Xm = Ym } x, y \in E. Let Z0 = 1 and Zn-1, i for n \in N. Then ϕ is an infinitely divisible CFP. 367 16.1 Lévy-Khinchin Formula.. Let Xn , n \in N, be Poisson random variables with parameters λn . L(ϕ) is nonempty and $\phi = \sup L(\phi)$. Define Sn = T1 + . (iii) A \in FA $\in \tau$ for any F $\subset \tau$. We show that X has a continuous modification on [0, 1]. Since f $\epsilon \uparrow f$ and $q \epsilon \uparrow q$ for $\epsilon \downarrow 0$, the monotone convergence theorem implies (4.8). For example, consider E = {0, . Define C := U $\cap Kn : U \in U$, $n \in N$ and C := 0 N 1 Cn : N $\in N$ and C1, . If $\phi: I \rightarrow R$ is a map, then we write $L(\phi) := g: I \rightarrow R$ is affine linear and $g \leq \phi$. Closed subsets of Polish spaces are again Polish. Then μ is an \emptyset -continuous content but not a premeasure. (17.13), (17.14) and (17.15) hold, then q is called the Q-matrix of X. Definition 16.3 The compound Poisson distribution with intensity measure $\nu \in Mf(R)$ is the following probability measure on R: CPoi $\nu := e * (\nu - \nu(R)\delta 0) := e - \nu(R) \propto *n \nu n = 0 n!$. $\Omega \times [0,T]) 3 \propto *$ Step 3. Takeaways A priori, checking equality of two measures by computing integrals or checking weak convergence of a sequence of measures requires to consider a huge class of test functions. A ring is called a σ -ring if it is also σ -U-closed. Furthermore, for all $r \in (1, s)$, 1 f (t) = $2\pi i \propto -\infty t - (r+i\rho) \phi f$ (r + i ρ) ϕf series in (3.1) converges for some z > 1, then the statement is also true for any $r \in (0, z)$ and we have $\lim \psi X(n)(x) = \psi X(n)(1) < \infty$ for $n \in N$. For $k \in K$, let $Bk \in Z k$ and $Jk \subset Ik$ be finite with $Bk = j \in Jk$ Aj for certain Aj $\in \sigma(Xj)$. Proof As $L(\phi) = \emptyset$ by Corollary 7.8, we can choose numbers a, $b \in R$ such that $ax + b \leq \phi(x)$ for all $x \in I$. As characteristic functions determine distributions, the claim follows by Theorem 13.34. n In other words, (PSn /n)n N atisfies an LDP with rate n and rate function I[~]. Then the following are equivalent: (i) X is aperiodic. For pairwise independent random variables with first moment, we could establish a strong law of large number via an involved truncation procedure which allows to use second moments estimates. We prepare for the proof of Theorem 16.5 with a further theorem. Submartingales are unfavourable games (the mean future is not as good as the present). $n \rightarrow \infty$ Show that μ is a measure on (Ω , A). n n-1k=0 " \leftarrow " Now assume that (20.7) holds. + We now construct Ω + with $\infty\phi(\Omega) = \alpha$. As one of the most prominent orders we present here the so-called stochastic order and illustrate its connection with CFP $\phi\theta$, r (t) = exp(r $\psi\theta$ (t)), where *n $\psi\theta$ (t) = log(2) $-it/\theta$), is infinitely divisible with $\Gamma\theta$, $r = \Gamma\theta$, r/n. The limits in (i) and (iii) coincide. By construction, $L = V \subset U$ is compact open set $L^{\circ} \supset K$. Define Y = (Y(1), B) assumption, $A \cap B \in D$, and trivially $A \cap B \subset A$. By the monotone convergence theorem (Theorem 4.20), we conclude $f d\lambda = \lim I n \rightarrow \infty I b gn d\lambda = f(x) dx$. Let Xt := YTt and Px = 0 PYx \otimes PT0. Theorem 1.61 (Finite products of measurable. " \leftarrow " Now let (X, (Px)x \in E) be a Markov process. We say that ν is singular to μ . Therefore, $f(y) \in B\delta(f(x)) \cap D = \emptyset$ and $f(z) \in B\delta(f(x)) \cap D = \emptyset$ and $f(z) \in \delta(f(x)) \cap D = \emptyset$ and $sup \omega \in A$ with $\mu(\Omega \setminus A) < \varepsilon$ n $\rightarrow \infty$ and sup $\omega \in A$ (fn (ω) $- f(\omega)$ $- \rightarrow 0$. 699 Name Index .. n n \in N n Definition 20.30 (Entropy of the simple shift) h(P, τ) is called the entropy of the dynamical system (Ω , A, P, τ)., $\beta n \in \mathbb{R}$ as well as d, $e \in \mathbb{R}$. Consequently, $\nu a \ 0 \ \mu$ and $\nu = \nu a + \nu s$ is the decomposition we wanted to construct. (i) (ii) (iii) (iv) N \mu 1, \sigma 2 * N \mu 2, \sigma 2 = N \mu 1 + \mu 2, \sigma 2 + \sigma 2 for $\mu 1$, $\mu 2 \in \mathbb{R}$ and $\sigma 12$, $\sigma 22 > 0$. Below the critical temperature, the magnetization increases with decreasing temperature. (3.9) In particular, we have ∞ 2n - n n $\sqrt{=4 \text{ x n 1} - \text{x n} = 0}$ 1 for all x \in C with $|\mathbf{x}| < 1$. Hence I1A is measurable for all A \in A1 \otimes A2. (23.23) for any β . Then τ is a bounded stopping time and sup Xm \geq a \Rightarrow m \leq n Let f (m, X) $= 1 \{m \le n\} 1 \{Xn - m > a\} + 1 2 \tau \le n.$ (iii) The distribution of τb has density fb (x) = $\sqrt{b} 2\pi e - b 2/(2x) x - 3/2$. Show that Xn $\rightarrow X$ if and only if D Xn + Yn $\rightarrow X$. Theorem 1.64 (Product measure, Bernoulli measure, Bernoulli measure) Let E be a finite nonempty set and $\Omega = E N$., μn be probability measures on R, B(R) Then there exists a smallest σ -algebra σ (E) with E $\subset \sigma$ (E): σ (E) := A. At this point, we only briefly make plausible the existence theorem for such regular versions of processes in the case of so-called Feller semigroups. variables with more general index sets are called stochastic processes. In other words, there are contents that are not premeasures. However, rigorous proofs are known only for d = 2 and $d \ge 19$ (see [67]). Then $f \circ \phi$ is bounded and measurable and Uf $\circ \phi \subset U\phi$; hence $\mu(Uf \circ \phi) = 0$. Consider the case $\Omega = R$, μ the Lebesgue measure and fn := 1[n,n+1], $f \equiv 0$., Ak for certain pairwise disjoint non-empty sets A1, By Theorem 16.5, we have CPoivn $\rightarrow \mu$. Consider a Poisson process (Nt) $t \geq 0$ and choose an independent exponentially distributed random variable T (it would suffice for T to have a density)., ωn] is the product of the probabilities of the individual events; that is, $\mu([\omega 1, .1044), ..1044)$ The Integral Exercise 4.1.1 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-x} \mathbb{1}[0,\infty)$ (x), and let λ the Lebesgue measure on \mathbb{R} . Now fix $\varepsilon > 0$. Show the following: (i) The effective conductance between x0 and x1 is Ceff (x0 $\leftrightarrow x1$) = d. Then $\mathbb{C} = \{f\lambda, \lambda \ge 0\}$ separates points, $f0 = 1 \in \mathbb{C}$ and $f\mu \cdot f\lambda = f\mu + \lambda \in \mathbb{C}$. In addition, these functions are exponentially scaled with $1/\epsilon$. n n-1 lim n $\rightarrow \infty$ k=0 (20.7) 20.5 Mixing 507 Proof " \Rightarrow " Let (Ω , A, P, τ) be ergodic. Exercise 2.1.1 In a queue each new arriving person chooses independently a random waiting position. Exercise 16.2.1 Let μ be an α -stable distribution and let ϕ be its characteristic function. L Thus, eventually there will be a consensus of all individuals, and the probability that the surviving opinion is $e \in \{0, 1\}$ is the initial frequency of opinion e. For any $x \in E$, dx := gcd(N(x, x)) is called the period of the state x. Indeed, for s < t and for the case of a submartingale, $E[Xt Fs] = E[E[Xt Fs] = Xs \cdot n \rightarrow \infty$ $n \rightarrow \infty$ $n \rightarrow \infty$ Hence Z = 0 and thus E[Zn F] - 0 almost surely. In particular, we get (as shown already in Example 4.22) that E[Sn] = 0 for all $n \in N$. We follow the exposition in [37]. \bullet Example 10.7 Let Y1, Y2, For $A \in A$, we define f du := A (f 1A) du. If we let $\varepsilon \to 0$, intuitively we should get the conditional probabilities as proportional to the thickness (in metres). Case 2: Var[Y] > 0. Note that Bn, $k \propto 2 - (n+1)/2$ if $n \in N$ and Bn,k Bn,l = 0 if k = l., DT. Remark 7.41 (i) If ϕ is a signed measure, then in (7.10) we automatically have absolute convergence. be identically distributed, real random p n-1 Xk for n \in N. Furthermore, E[Yn (x)] = P[Xn < x] = F (x) and E[Zn (x)] = P[Xn < x] = F (x) 1.19 yields $A \supset DE \supset \delta(E) = \sigma(E) = A$. 14.4 Markov Semigroups. We define a set function $\mu^{\sim} F : A \rightarrow [0, \infty)$, (a, b] $\rightarrow F(b) - F(a)$. \diamond Theorem 2.26 Let K be an arbitrary mutually disjoint index sets. Hence, by Theorem 12.14, *) $n \rightarrow \infty$ An (ϕ) $- \rightarrow E \phi(X)$ E a.s. and in L1. be independent random variables with P[Xn = 1] $\in (0, \infty)$. 1) for all n \in N. Klenke, Probability Theory, Universitext, 493 494 20 Ergodic Theory Example 20.3 (i) If X = (Xt)t \in I is i.i.d., then X is stationary. It suffices to consider B = (Bt)t \in [0,1]. Then a.e. convergence on each An. (For example, for any x \in K, take an open ball Bex (x) of radius ex > 0 that is conta that is relatively compact. $k \rightarrow \infty \ N \rightarrow \infty$ On the other hand, Acn, $N \downarrow \emptyset$ for $N \rightarrow \infty$; hence $\mu(Acn, N) \rightarrow 0$. $n \rightarrow \infty$ In both cases, we have fn $-\rightarrow f$ almost everywhere. 107 5 Moments and Laws of Large Numbers . By Theorem 15.57, this yields the claim. In statistical physics, a key quantity is the so-called partition function
$Zn\beta := e - \beta Un \ d\mu 0n$. More precisely, there exists a unique finite C for any $B \in Fn$. Indeed, $\# \in S(n) : -1$ (i) $\leq l$ for some $i \in \{1, . If \lambda n (A) is finite, then for any <math>\epsilon > 0$ there exists a compact $K \subset A$ such that $\lambda n (A \setminus K) < \epsilon$. random nt ! n For t > 0, let St = i = 1 Yi and $Stn = \sqrt{12}$ Stn. By assumption, the set $E := \{Xi - 1 (A) : A \in Ai , i \in I\}$ is a generator of A. Hence, it is enough to show that the integrals along δb ,t and # c,t vanish if $b \to 0$ and $c \to \infty$. $\infty \propto \mu(An) \leq \mu$ An for any choice of countably many (iv) If A is a ring, then $n=1 \ n=1 \ \infty$ mutually disjoint sets A1, A2, . We study periodicity of Markov chains in the first section. $\sigma > \tau$ is a stopping time In fact, obviously, we have $F\tau n \supset F\tau + for$ all n., $\omega n \in E$ and $n \in N$. (8.10) If X is surjective, then ϕ is determined uniquely. (For example, for the Petersburg game (Example 4.22) we had Fn (x1, . Hence, in the following, assume g(t) < ∞ for all t > 0. 22.3 Hartman-Wintner Theorem . Let N = { ω : f (ω) > 0}. \in M \leq 1 (R) with corresponding distribution functions F, F1, F2, . By construction, A is predictable with A0 = 0, and M is a martingale since *) E[Mn - Mn - 1 Fn - 1] = E Xn - E[Xn Fn - 1] Fn - 1 = 0. $n \rightarrow \infty$ (ii) (Xn, 1) is a null array and PSn $\rightarrow N0, 1$. $n=1 k \rightarrow \infty$ Then $\alpha k \epsilon = 2-k 3 \infty 0 \infty q \epsilon$ (t) dt. If $r \in F$ is an arbitrary fixed point of ψ , then $r \ge 0 = q0$. Later we will come back to the introductory example and make the computations explicit. Hence we only have to show that the conditional distributions exist. The starting point will be to define the values of u on a smaller class of sets; that is, on a semiring. However, in practice, this distinction will not be needed in this book, we have Ei = E Takeaways Assume that a Markov chain can return to a given state only at times that are a multiple of some natural number d and assume that d is the largest number with this property. Reflection Find an example that shows that in (iii), we cannot simply drop the assumption that there exists a sequence En $\uparrow \Omega$ with $\mu(En) < \infty$. For $n \in N$, define the polynomial fn by fn (x) := n f (k/n) k=0 n k x (1 - x)n-k k for $x \in [0, 1]$. Then, in particular, $A \in F$; hence)*) *) * 1A E[E[X | F]|G] = E 1A E[X | F] = E[1A X] = E 1A E[X | G]. + R(5, 6). B $\subset U$:= n=1 Un. Find an example that shows that this assumption cannot be dropped. Then D(ϵ) := ∞ n=1 Dn (ϵ) $\subset N$, where N is the null set from the definition of almost everywhere convergence. By Theorem 1.18, it is sufficient to show that M(μ *) is a λ -system. Now let $\Omega 1$, $\Omega 2$, Definition 8.35 A measurable space (E, E) is called a Borel space if there exists a Borel set B \in B(R) such that (E, E) and (B, B(B)) are isomorphic. Then $\nu(An) < \infty$; hence (f1 – f2) d μ . Let I be the unit matrix on E. Roughly speaking, the difference is that vague convergence does not imply convergence of total masses. Theorem 5.32 (Baum and Katz [8]) Let y > 1 and let X1, X2, Then the canonical process Xn : (Rd) N0 \rightarrow Rd is a Markov chain with distributions (Px) $x \in \mathbb{R}^d$. Clearly, this measure is σ -finite; however, it is neither locally finite nor outer regular. Definition 9.37 (Discrete stochastic integral) Let (Xn) $n \in \mathbb{N}$ be a realvalued and F-predictable process. + Xk for k = 1, (21.4) Hence $s \rightarrow t Xs - xt$ in probability. Then $f = f + -f - with f + , f - \ge 0$ being integrable functions. c1 (ek), PX is equal to that X0 = x0 and Xn = 1, Let (Ye) e $\in E$ and $q \in \{p, p, 1\}$. = fn (X1, . Hence d(B1, Ac1) > 0. = Z0i = 1, then D Z = Z1 + . This implies Sn \geq S + 2 for n \geq n0 and hence lim infn $\rightarrow \infty$ Sn /n \geq pc 2 > 0. Sometimes we will drop the qualifications " almost surely ". Hence (B(R)) \otimes [0, ∞) Ω = σ Xt, t \in [0, ∞) = B(Ω , d). The subset of probability measures is denoted by M1 (Ω) := M1 (Ω , A). Then f is Hölder-continuous of order γ with constant C := C(ϵ) "T / ϵ #1- γ . By the dominated convergence theorem, we get (25.3). Then ϕ n = ϕ 1/n serves the purpose. Let Z 1, . We denote the equivalence relation by R and let UL = KL /R be the set of equivalence classes. Then α f + β g is also harmonic on E \ A. Definition 9.22 If $\tau < \infty$ is a stopping time, then we define $X_{\tau}(\omega) := X_{\tau}(\omega)(\omega)$. In the latter case, the absolute value of the mean magnetization β_{0} β_{0} is $m \pm = m + > 0$. Definition 7.4 A subset G of a vector space (or of an affine linear space) is called convex if, for any two points x, $y \in G$ and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in G$. Then $E[X] = n \ kP[X = k] = k = 0$ n n k k p (1 - p)n - k $k = 0 = np \cdot n$ n-1 k=1 k-1 pk-1 (1 - p)(n-1)-(k-1) = np. (ii) (Stability under complements) If $A \in A0$, then X-1 ((A) $c \in \sigma$ (X-1 (E)); hence (A) $c \in A0$. (21.24) Lévy Construction of Brownian motion. The main theorem says that any function of the history up to a given time t can be represented as a discrete stochastic integral with respect to this binary splitting process. and Y1, Y2, If $G \subset F$ is a σ -algebra, then we write L[X | G] for the regular conditional distribution of X given G. Here one choice (and thus up to multiples the unique choice) for the conductances is C(x, x + 1) = p 1 - p x for $x \in Z$, and C(x, y) = 0 if |x - y|> 1. Letting $\varepsilon \to 0$, we get (13.14). Our next goal is to deduce simple criteria in terms of distribution functions and densities for checking whether a family of random variables is independent or not. By (i), we have (P3). Thus (wn)n \in N is a Cauchy sequence: wm – wn – $\to 0$ if m, n $\to \infty$. If this is the case, then strict inequality holds if H (p) < ∞ . 24.1 Random Measures 615 independent increments and that PX(A) = PX1 (A) * PX2 (A) * . To this end, we distinguish two cases. Klenke, Probability Theory, Universitext, 163 7 Lp -Spaces and the Radon-Nikodym Theorem 164 the triangle inequality, to this end, we have to change the space a little bit since we only have f - gp = 0 = 0 = f = g µ-a.e. For a proper norm (that is, not only a seminorm), the left-hand side has to imply equality (not only a.e.) of f and g. Consider the distribution $\mu\alpha$ on R with density f α (x) = 1 |x|-1-1/\alpha 1 {|x|>1}. Clearly, $E \subset DE$; hence $\delta(E) \subset DE$; hence with $\Omega \in \text{Ei}$. In the rest of this chapter, we let (Ω , A) and (Ω , A) be measurable spaces. Starting d with a graph other than Z, for example an infinite binary tree, can result in multiple infinite connected components (Exercise 2.4.1). We can argue as follows. be exchangeable, square integrable random variables. Hence (16.4) holds if $n \rightarrow \infty$ the array (Xn, 1) is a null array. If in particular $\sigma(E) = A$, then X is A - -A-measurable $\iff X-1(E) \subset A$, the I, we have that (Xt1, .281 Basic Measure Theory As in Example 1.54, it can be shown that $\mu^{\sim} F((a, b)) \le \epsilon + \infty^{\sim} F((a(k), b(k)))$. $= X0 + Z1 + ..., n\} \ge t \le t - 2$ Var[Sn]. As soon as constructing probability spaces has become routine, the concrete probability space will lose its importance and it will interest us. Example 17.55 Let X be a real random variable and let f, g: R o R be monotone increasing functions with E[f(X)2] < \infty and E[g(X)2 $| < \infty$. (i) Show that I - p is invertible. For x = (x0, . 6.1 Almost Sure and Measure Convergence. Hence, for any $\varepsilon > 0$, there exists a compact set K \subset E with P[X1 \in K c] $< \varepsilon 2$. Furthermore,) 2 * n $\rightarrow \infty$ Ln (ε) = E Y1 1{|Y1| > $\varepsilon \sqrt{n}$ } $\rightarrow 0$; hence (Xn,I) satisfies the Lindeberg condition. Hence ∞) * $\sigma 2 \psi(t) := \log E$ eit X = -t 2 + ibt + ψk (t) 2 k=0 satisfies the Lévy-Khinchin formula $\sigma_2 \psi(t) = -t 2 + ibt + 2$ eit $x - 1 - itx 1\{|x| 0, n \in N$ Similarly, we get $\mu(\{f1 < f2\}) = 0$; hence $f1 = f2 \mu$ -a.e. Definition 7.30 Let μ and ν be two measures on (Ω, A) . XN = xN] = P[X1 = x1] N-1 P[Xk+1 = xk+1 |X1 = x1]. -algebra of the σ -past), PI [X \in A F σ] = PX σ [X \in A]. For d \geq 2, we have pc (d) \in 2d-1, 3. Hence P[Ai] = whence +, PAj = P[Aj]. Proof (i) For t \in I, we have $\{\sigma \lor \tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land
\tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \leq t\} = \{\sigma \leq t\} \cup \{\tau \leq t\} \in$ Ft and $\{\sigma \land \tau \in t\} \in$ Ft and $\{\sigma \land \tau \in t\} \in$ Ft and $\{\sigma \land \tau \in t$ expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. In particular, it is analytic and is hence determined by the coefficients of its power series about t = 0; that is, by the moments of X. sup $f \in F(6.5)$ { $|f| > g_{\epsilon}$ } Clearly, (ii) implies (i). " \Rightarrow " By the usual approximation arguments, it is enough to consider functions f that depend only on finitely many coordinates $0 \le t1 \le t2 \le$. David Wilson has nice simulations and a survey of the current research on his web site . n +1 wLn + wn-Ln Hence (Xn)n \in N0 is our generalized urn model with weights (wn)n \in N0. Finally, in the third step, an embedding of E into RN is constructed. Hence) 1 E X¹] = $\phi^{(0)} = \phi(\tau) = 0$,) 1 Var X¹] = $\phi^{(0)} = \phi(\tau) \in (0, \infty)$. Corollary 21.32 The map F1 : $\Omega \to [0, \infty)$, $\omega \to \sup\{\omega(t) : t \in [0, 1]\}$ is A-measurable. (ii) If $f d\mu < \infty$, then $f < \infty$ almost everywhere. Since every CPoivn is infinitely divisible, on the one hand we have to show that this property is preserved under weak limits. "(iii) $\Rightarrow (iv)$ ' Let A1, A2, This transition matrix is not irreducible; rather it has two absorbing states 0 and N. 10.2 Optional Stopping. Then F -1 (t) $\leq x \iff t \leq F(x)$. If, in addition, q is aperiodic, or q is not reversible with respect to π , then p is aperiodic. Now we can drop the quotation marks from the statement and write it down formally. Define $J = k \in K Jk$. That is, we denote the edge that connects x and y by)x, $y^* = y$, x^* instead of $\{x, y\}$. Clearly, we have $E[E[|X| \land N | F]_2] \le N 2$. That is, let Xn, i , n, i $\in N0$ be i.i.d. random variables on N0 with P[Xn, i = k] = p(k), $k \in N0$, and based on the initial state Z0 define inductively Zn+1 = Zn Xn, i . 18.4 Speed of Convergence So far we have ignored the question of the speed of convergence of the distribution PXn to π . Proof See, e.g., [83, page 79]. 2 h \in Z earranging this formula yields an expression for the number of leaves: + # u \in Z : degHL (h) \geq 3 \geq 2 + # (Z \cap TL).., then X = ∞ X has intensity measure E[X] = μ and hence X is n=1 n a random measure (see Exercise 24.1.1). Show that $n \rightarrow \infty \nu t / n \rightarrow \delta 0$. ($n \rightarrow \infty = 0$, For any $\varepsilon \in (0, 1]$, we have $\varepsilon \rho C, \varepsilon \in Lip1$ (E; [0, 1]). 6.1 Implications between the concepts of convergence. In the general case, it follows by Theorem 20.21., hzn). \bullet 8.3 Regular Conditional Distribution 211 Exercise 8.3.3 Assume the random variable (X, Y) is uniformly distributed on the disc $B := \{(x, y) \in \mathbb{R}2 : x 2 + y 2 \le 1\}$ and on $[-1, 1]^2$, respectively. In this section, we show how to do this. A subset $C \subset Cb$ (E; K) is called an algebra if (i) $1 \in C$, (ii) if $f, g \in C$, then $f \cdot g$ and f + g are in C, and (iii) if $f \in C$ and $\alpha \in K$, then (αf) is in C. A simple algorithm for this method is the following. Show that X is null recurrent, irreducible and aperiodic and that independent coalescence does not give a successful coupling. Let W1 := U1. By considering Laplace transforms, we obtain that, for every $\lambda \ge 0$, the sequence of distributions converges: $n \mapsto \infty = \lim n \to \infty \psi(t n!) + 1 - nt$ $e - \lambda/n = \lim 1 - n \rightarrow \infty$ ($e - \lambda/n$) nx $1 - e - \lambda/n$ n($1 - e - \lambda/n$) t + 1 nx (21.44) = exp - \lim x n(1 - e - \lambda/n) t + 1 \lambda (x/t) := $\psi t (\lambda) x$. Interpret the statement of Theorem 23.11 in this case. 38 1 Basic Measure Theory Corollary 1.82 (Measurability of composed maps) Let I be a nonempty index set and let (Ω , A), (Ω , A) and (Ω i, A) and (Ω i, A) be measurable spaces for any $i \in I$. Then F (0) = 0 since F is linear. The subsequent section proves the CLT for real-valued random variables by means of characteristic functions. Then Ntn ~ b2n, pn . Hence P[X ≥ n+1] = (1-p)n - (1-p)n + 1 = p(1-p)n - (1-p)n + 1 = p(1-p)n - (1-p)n + 1 = p(1-p)n + probability 1/2D. n=m In the last step, we used Remark 7.41(i). 3.2 Poisson Approximation .. Letting $\epsilon := \epsilon \operatorname{Var}[Sn]$, we get Ln ($\epsilon) \le \epsilon - \delta \operatorname{kn}^*$) 1 E [Xn,l |2+ δ . Corollary 16.9 If (μ n)n \in N is a (weakly) convergent sequence of infinitely divisible probability measures on R, then $\mu = \lim_{n \to \infty} \mu n$ is infinitely divisible. To put it differently, under Px, the process (Bt - x)t ≥ 0 is a standard Brownian motion. For a random walk on Z with a finite first moment, this shows that it is recurrent if and only if the increments are centred. We infer x, y $\in E = J(x, y) + D(x, y) R(x, y) + 2I(x, y) + D(x, y) R(x, y) + 2I(x, y) + D(x, y) R(x, y) R(x, y) + 2I(x, y) + D(x, y) R(x, y) R(x, y) + 2I(x, y) + D(x, y) R(x, y) R(x, y) + 2I(x, y) R(x, y) R($ $v_{1} + 2 u(x) - u(y) D(x, y)$, (iii) Let $n \in N$ and A. A1 ... Xd)T is called d-dimensional normally distributed with expectation u and covariance matrix C if X has the density fu. C (x) = 2 1B C exp - x - u. C - 1 (x - u) 2 (2\pi)d det(C) 1 (15.10) for $x \in Rd$, 2 x 27/5 0 x R (x, 1) = 27/26 1 1 R (0, x) = 27/32 54/25 513/125 19 R (0, 1) = 27/8 0 Fig. Of course, two independent chains form a coupling, though maybe not the most interesting one. Then ϕ is concave (exercise!); hence, for nonnegative random variables X and Y with finite expectation (by Theorem 7.11), ' (E X α Y 1- α < (E[X]) α (E[Y])1- α . Let (Yn) n \in N0 α is concave (exercise!); hence, for nonnegative random variables X and Y with finite expectation (by Theorem 7.11), ' (E X α Y 1- α < (E[X]) α (E[Y])1- α . Poisson process with rate λ . (ii) Compute pmax explicitly. On the trace σ algebra A, we define a measure by Ω μ (A) := $\mu(A) \Omega$ for $A \in A$ with $A \subset \Omega$. 5.3 Strong Law of Large Numbers 129 Assume that the computer generates numbers X1, X2, . in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use. Further, let F0 + = + t > 0 Ft. Then we have $n \rightarrow \infty \sqrt{2}$., An \in A such that $\mu(A1 \cup . In the first round, the stake is H1 = 1$. Proof Let $\lambda \in \mathbb{R}^d$. Only for the more general case of stationary $n \rightarrow \infty 1$ X do we need an additional argument.) By the ergodic theorem, we have n Sn $\rightarrow \rightarrow$ E[X1 I] = 0 a.s. Thus, for every m \in N, lim sup n $\rightarrow \infty$ 1 1 max Sk = lim sup max Sk n k=1,...,n n $\neq \infty \leq$ max k \geq m [Sk | m $\rightarrow \infty \rightarrow 0$. Show that X and Y are independent. Hence we only have to show sufficiency of the two conditions. Example 4.22 (Petersburg game) By a concrete example, we show that in Fatou's lemma the assumption of an integrable minorant is essential. Second compactness. - Frequency 0.0651 0.0189 0.0306 0.0508 0.1740 0.0166 0.0301 0.0476 0.0755 0.0027 0.0121 0.0344 0.0253 Letter N O P O R S T U V W X Y Z Morse code -. e ∈ E ♦ Example 20.32 (Markov chain) Let (Xn)n ∈ N0 be a Markov chain on E with transition matrix P and stationary distribution π. Let A ∈ I (recall that I is the invariant σ algebra) and B = A. are independent Poisson point processes with intensity measures μ_1 , μ_2 , Furthermore, let Z1, Z2, \bullet Remark 9.27 If I = N, I = N or I = Z, then it is enough to consider at each instant s only t = s + 1. That is, those x for which $\varphi(x) - I(x)$ is close to its maximum. On the other hand, A is called sequentially compact (respectively relatively sequentially compact) if any sequence (xn)n \in N with values in A has a subsequence (xn)k \in N (respectively x \in A). Corollary 16.10 If $\mu \in$ M1 (R) is infinitely divisible, then there exists a continuous convolution semigroup (μ t) $t \geq 0$ with values in A has a subsequence (xn)k \in N (respectively x \in A). stationary increments $Xt - Xs \sim \mu t - s$ for t > s. Let $g: A \rightarrow R$ be a bounded function. It is an interesting finding that in two important examples we could check σ -subadditivity using topological properties. In order to simplify the notation, we may assume that X is the canonical process on E N0. $n \rightarrow \infty$ We write $\mu n \rightarrow \mu$ if any of the four conditions holds and say that (µn) $\in N$ converges weakly to µ. 20.5 Mixing 509 " = " Let X be periodic with period d ≥ 2. In particular, if A = $\times j \in J$ Aj for certain 306 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on
Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$, then XJ-1 (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures on Product Spaces Aj $\in Aj$ (A) is called a rectangular cylinder with base J. k=1 314 14 Probability Measures , A0), then $\mu i := \mu \otimes \kappa k$ is a finite measure on k=1 i i / $\times \Omega k$, Ak . ($-\infty, x$] Here ($-\infty, x$] + { $y \in Rn : yi \le xi$ for i = 1, . Reflection Come up with an example for X such that the series in (3.1) does not converge for any z > 1 but lim $\psi X(x)$ exists and is finite. (i) Compute the generating function ψ and the extinction probability q. Theorem 21.6 yields the existence of a version B of X that has Hölder-y -continuous paths. In particular, here $\alpha = \infty$. If $\mu(\Omega) < \infty$, then (i) implies uniform integrability of F since the infimum is taken over the smaller set of constant functions. While the preceding corollary only yields an abstract uniqueness statement, we will profit also from an explicit inversion formula for

Fourier transforms. (21.39) 554 21 Brownian Motion We consider now the probability generating function of X1,1, $\psi(1)(s) := E[s X1,1]$, $s \in [0, 1]$. is independence of classes of events) Let I be an arbitrary index set and let Ei \subset A for all $i \in I$., Xn be integrable i.i.d. random variables. Lef hand side: n = 1000, Right hand side n = 10 000. Hint: Use suitable stopping times K and apply the martingale convergence theorem (Theorem 11.4) to the stopped process XK. Intuitively, this is the symmetric simple random walk whose vertical transitions are all blocked away from the vertical axis. The next theorem shows that if the Xn are integrable, then the process of partial sums can go to infinity only with a linear speed. 21.1 Computer simulation of a Brownian motion. Hence, it remains to show measurability of If . - Z n $-(n+1) \otimes *$) E 1{Zn =k} k · Xn, i Fn k=1 = m - n $\infty *$) E k · 1{Zn =k} Fn k=1 = m - n Zn = Wn . Note that T C E; hence the statement is trivially true if the roles of E and T are interchanged. Let $W := (W1, The discrete stochastic process H \cdot X defined by (H \cdot X)n := n$ Hm (Xm - Xm-1) for $n \in N0$. 552 21 Brownian Motion We apply this twice (with a = (t + s)n - (t + s)n! and a = "sn# - sn) and obtain (using the rough estimate " $(t + s)n\# - sn! \le tn + 2 \le 3tn$ from (21.36) (since $t \le N$) *) Kn Kn 4 * E ($T^{-}tK + sn, n - T^{-}sKn, n$) 4 $\le n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) Kn = $n - 2\sigma - 4 \ge (T^{-}(t + s)n\# - T sn!)$) absolutely. & Chapter 7 Lp -Spaces and the Radon-Nikodym Theorem In this chapter, we study the spaces of functions whose pth power is integrable. A family $F \subset Lp$ (μ) is called bounded in Lp (μ) if sup{f $p : f \in F$ } < ∞ . A fundamental question is: When does a sequence (μ) $n \in N$ of measures on (E, E) converge weakly or does at least have a weak limit point? Show that the reverse inclusion to Theorem 4.19 holds, Lp (μ) \subset Lp (μ) if $1 \leq p \leq \infty$. You can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywords or location and, sometimes, you can search these sites by name, keywo also part of the company's offerings. The additional claim follows by the Portemanteau theorem (Theorem 13.16) since N0,1 ($\partial[a, b]$) = 0. Thus X0 \geq Sk+1 – Mn $\circ \tau$ for k = 1, . Clearly, Hn may only depend on the results of the gambles that happened earlier, but not on Dm for any m \geq n. By Lemma 15.23, this implies uniform convergence on compact sets.) with $P[Zi = i] = P[Zi = -i] = 1 \ 11 \ ...$ Let $E \in E$ with $u(E) < \infty$. However, p is sufficiently contractive only if the multiplicity of the eigenvalues of modulus 1. 19.6 Random Walk in a Random Environment . 24.1 Random Measures 613 Theorem 24.5 Let PX be the distribution of a random measure X. If 3E[X] = 0, then X is called centered. Then, by construction, P[Rn(i) = j] = r(i, j) - r(i, j - 1) = p(i, j). are i.i.d. E-valued random variables with P[Xi = e] = pe for $e \in E$. Now $A0 = \sigma(E)$ since $E \subset A0$. Before we show that Cramér's theorem is essentially an LDP, we make two technical statements. Let $h = 0 e - t \kappa t g dt$. n = 1 Reflection In the above theorem, why did we need that μ is finite? Let X be a Markov process with transition kernels (κt) $t \ge 0$ and with respect to a filtration F that satisfies the usual conditions. Proof We explicitly construct a probability space (Ω , A, P) and a random variable X : $\Omega \rightarrow R$ such that FX = F $(c) (-f) d\mu = -f d\mu$. Let $n \in N$ with $C \subset Kn$. We have thus reduced the problem to the one-dimensional situation and will henceforth assume d = 1. By Lemma 20.7, we have $E[X0 | I] \circ \tau = n := Xn - E[X0 | I]$, without loss of E[X0 | I] Pa.s. Hence, by passing to X generality, we can assume E[X0 | I] = 0. In order to work with the concepts of weak convergence in this proof, we introduce the function $\begin{bmatrix} 1 & \text{if } x = 0, \\ 1 & \text{if$ s is a stopping time. By concatenation of stochastic kernels. In particular, for $X \in L1$ (P), the family (E[X |Fj], $j \in J$) is uniformly integrable., Ajn-1 \in B(E) and A := $\times j \in L$ Aj. 23.2 The shifted free energy F β (m) – F β (0) of the Weiss ferromagnet without exterior field (h = 0). Hence we expect the convergence in the central limit theorem to be slower. Recall the definition 1.25. + Xn)2k - k E X12 nk ≤ d2k nk-1. While here this is only of interest in that it simplifies the computation of fair prices, it has an economic interpretation as a measure for the market prices that we would see if all traders were risk-neutral; that is, for traders who price a future payment by its mean value. If $X \sim N\mu$, C, then the following statements hold. Let E = N and $\begin{bmatrix} 2 \\ 4 \end{bmatrix} x$, q(x, y) = -x 2, $\begin{bmatrix} 0, & \text{if } y = x + 1, &$ δxi, where g has to be chosen appropriately (depending on f)., ω] = pωi. Use Doob's inequality (Exercise 21.4.1) to show that the martingale convergence for uniformly integrable martingale convergence theorem (Theorem 11.10)) hold for X. Example 17.53 Let (px) $x \in N0$ be numbers in (0, 1] and let X be an irreducible Markov chain on N0 with transition matrix p(x, y) = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (px , 1 - px , 0, if y = x + 1, if y = 0, else. Example 14.33 We come back to the example from the beginning of this chapter. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations. Since the distribution of (X1, . Hence $\mu * (E \cap A) + \mu * (E \cap Ac) \le \mu * (E)$ and thus $A \in M(\mu *)$, which implies $A \subset M(\mu *)$. Let g := |f|(q-1). Hence by the induction hypothesis, we have $P[Nn1, t+1 \ge l] = E[hn1, l(Nn2, t)] \le E$ the assumption (16.30), we have $\alpha < 2$. 20.3 Examples . As we saw in Theorem 17.17, by Xn+1 := Rn (Xn) and Yn+1 := Rn (Yn), two Markov chains Sets of finite measure (of μ). / (i) Aj = $\sigma \times Ej : Ej \in Ej \cup {\Omega j}$ for every countable $J \subset I$ initial value X0 = x by defining $Xn = Fn \circ Fn - 1 \circ \cdots \circ F1$ (x). Finally, let $A := \{Sn = 0 \text{ for every } n \in N\}$ be the event of an "escape" from 0. On C([0, 1]) the norm $f \infty = supx \in [0,1] | f(x)|$ induces a topology. As compact sets are totally bounded, there exists an $N \in N$ and points t1,., $kD \in N0$ with k1 + .. n Hence the upper bound in the LDP holds (even for arbitrary A). Define the measure ν by $\nu(A) :=$ for $A \in A$. In this case, we say that $\mu 2$ is stochastically larger than $\mu 1$. We give just two different possibilities: $\mu = 12 \delta 1 + 12 \delta 3$ and $\mu = 12 \delta 2 + 12 \delta 4$. 3.3 Branching Processes . Similarly, X is a submartingale if E[Yt] ≥ 0 for all t, and a supermartingale if E[Yt] ≤ 0 for all t. Do it! (iii) Compute $\lim n \to \infty \psi n(z), z \in [0, 1]$. Then we clearly have LaX+b (t) = e-bt LX (at) and LX+Y (t) = LX (t) LY (t) for t \ge 0. ... Then: (i) (Linearity) $E[\lambda X + Y | F] = \lambda E[X | F] + E[Y | F]$. Remark 17.2 If E is a countable space, then X has the Markov property if and only if, for all $n \in N$, all s1 < . Now choose L large enough for P[A2L, 0] > 0. (ii) For X as in (i) with E[X]= 0, infer that (using Jensen's
inequality) $\geq 2 E e \lambda \leq \cosh(\lambda) \leq e \lambda / 2$ for all $\lambda \in \mathbb{R}$. \bullet Lemma 20.4 If (Xn)n $\in \mathbb{N}0$, we have E[Xs+1 Fs] = 0, infer that (using Jensen's inequality) $\geq 2 E e \lambda \leq \cosh(\lambda) \leq e \lambda / 2$ for all $\lambda \in \mathbb{R}$. E[Xs Ys+1 Fs] = Xs E[Ys+1 Fs] = Xs E[Ys+1 Fs] = Xs. Hence any countable union of sets. We saw that with probability one there is at least one infinite open cluster. Let $pN = \{p \in P : p \le n\}$. Proof (i) and (ii) follow by simple computations. We now show by elementary means the validity of (1.13). As r is irrational it is easy to see that A is generated by $n \in N$) and with the product measure (or Bernoulli measure) $n \otimes N$) * $P = p \delta$; that is where $P[\omega, .., An]$, the random variables X(A1), . a I Proof Choose t such that (4.6) holds. However, the Morse code also consists of gaps of different lengths that signal ends of letters and words. On the other hand, for any $f \in L2(\mu)$, the map $L2(\mu) \rightarrow R$, $g \rightarrow f$, g^* is continuous and linear. Then X is a martingale. be i.i.d. real random variables with λn (Bi \ Ai) < $\epsilon 2 - i - 1$ (upper semicontinuity of λn). Indeed, $n \rightarrow \infty$ by the triangle inequality, $f - g1 \leq fn$ $-f_1 + f_n - g_1 \rightarrow 0$. To this end, we agree on the following conventions. Show that P[X < Y] = θ . Definition 21.35 We say that the finite-dimensional distributions of (Xn) converge to those of X if, for every $k \in N$ and t1, Finally, we show that closed sets are measurable with respect to this outer measure. We first show that the matrix p is irreducible. Exercise 8.3.1 Let (E, E) be a Borel space and let μ be an atom-free measure (that is, $\mu(\{x\}) = 0$ for any $x \in E$). (iii) A is a σ -algebra if and only if $\sigma(A) = A$. For two Lebesgue integrable maps f, g : Rn $\rightarrow [0, \infty]$, define the convolution f * g : Rn $\rightarrow [0, \infty]$, by (f * g)(x) = Rn f (y) g(x - y) \lambda n (dy). In many cases, it is necessary to rescale the original distributions in order to capture the behavior of the essential fluctuations, e.g., in the central limit theorem. We thus obtain the Cox-Ross-Rubinstein formula $\pi(VT) = x0$ bT, $p(\{A, .., Yn) = (X1, X2/2, X3/3, .., Wn + Rw By symmetry (and since X is transient), we get)) * * n \rightarrow \infty$ P0 Xn $\rightarrow -\infty = 1 - P0$ $2 \text{ k! } \sqrt{(iii)}$ Let Sn* = (X1 + . Furthermore, XN \uparrow |X| and YN \uparrow |Y|. Show that F is tight if and only if L is bounded. We are now at the point to use a Markov chain. Theorem 19.7 (Uniqueness of harmonic functions) Assume that F (x, y) > 0 for all x, y \in E. Then f dµn \leq Fn (y0) + Fn (∞) - Fn (yN) + N (f (yi) + ϵ)(Fn (yi) - Fn (yi-1)). (i) (ii) (iii) (iv) (v) μ is σ -additive (and hence a premeasure). \in Mf (E). One sufficient condition for M < ∞ is α -1 exp - (1 - pk) n=0 k=0 < ∞ . 1.5 Random Variables.. Then, by Theorem 12.24, there exists a σ -algebra A \subset F such that (Xn) n \in N is i.i.d. given A. , ∞ } the (random) number of infinite open clusters. Now let $\nu n = 1R \{0\}$ nµn . Exercise 13.2.1 Let dP be $\sqrt{}$ the Prohorov metric (see (13.4) and Exercise 13.2.1). For any $\epsilon > 0$ (and $g\epsilon$ as above), 3ϵ choose at such that $\{g g\epsilon/2 > a\epsilon\} \epsilon/2 d\mu < 2$. $n \rightarrow \infty$ Since $\nu \in A$ is arbitrary, we get lim supn $\rightarrow \infty$ inf Iµ (A \cap En) = inf Iµ (A). Proof (i) This is immediate from the definition of the exercise 13.2.1). integral. n=1 As a sum of convex functions, H is convex. n $\rightarrow \infty$ This shows that almost surely Stn +1 < L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and this in turn contradicts the assumption that L be finite. 17.2 Discrete Markov Chains: Examples 401 Theorem 17.17 (i) With respect to the distribution (Px)x \in L - ϵ infinitely often and the distribution (Px)x \in L - ϵ infinitely often and the distribution (Px)x \in L - ϵ infinitely often and the distribution (Px)x \in L - ϵ infinitely often and the distribution (Px)x \in L - ϵ chain with transition matrix p. On the other hand, we write "f \leq g almost everywhere" if the weaker condition holds that there exists a μ -null set N such that f (ω) \leq g(ω) for any $\omega \in$ N c . 5.3 Simulation of a Poisson process with rate $\alpha = 0.5$. 6 5 4 3 2 1 0 0 2 4 6 8 10 5.5 The Poisson Process 141 Theorem 5.34 If (NI , I \in I) has properties (P1)-(P5), then $(N(0,t], t \ge 0)$ is a Poisson process with intensity $\alpha := E[N(0,1]]$. By E, R Theorem 1.81, we have $\sigma Z\{i\} = \sigma(Xi)$ for all $i \in I$; hence $\sigma(Xi) \subset \sigma(Z \in R)$. probability f (X1)/c and is rejected otherwise. In the above description, we have constructed by hand a two-stage experiment. Clearly, h is continuously differentiable on R. Now let $D = A \in A1 \otimes A2$: I1A is A1 -measurable. In the particular case V = L2 (µ), by the Cauchy-Schwarz inequality, we have F 2 = f 2. $\in B(E)$ with $An \uparrow A$. $\in Lp$ (Ω , A, P) with $Xn - Xp \rightarrow 0$, ;; ; E[Xn | F] - E[X | F]; $n \rightarrow \infty \rightarrow 0$., Xn-1 = xn-1] = P[Yn-1,1 + .18.1 and 18.2 for illustrations of aperiodic Markov chains. If ϕ is the characteristic function of the measure μ from the previous example, then clearly $\psi(t) = |\phi(t)|$. The chain starts at 2, and we want to compute the probability that it visits 3 before 5. For general education we present Bochner's theorem that formulates a necessary and sufficient condition for a function ϕ : Rd \rightarrow C to be the characteristic function of a probability measure. For all $n \in \mathbb{N}$, we have i=1 $\mu * (E \cap Bn+1) \cap Bn + \mu * (E \cap Bn+1) \cap$ n) $k-1 Pk [\tau n < \tau 0] = u(k) = N n-1 l=0 R(l, l + 1)$. (ii) X is a submartingale (supermartingale) if and only if $H \cdot X$ is a submartingale) for any locally bounded predictable $H \ge 0$. (7.11) Indeed, letting $E := \{f \ge g\}$, for all $A \in A$, we have $(f \lor g) d\mu = A f d\mu + A \cap E g d\mu \le \nu(A \cap E) + \nu(A \setminus E) = \nu(A)$. Exercise 20.6.2 Let (an) $n \in N$ be a sequence on nonnegative numbers. Now the idea is to construct a Markov chain X whose distribution converges to π in the long run. However, this is exactly (21.3) with T = 1. Nevertheless, the eigenvalues and eigenvectors are of the same form as in Case 1. This is particularly helpful in the context of statistical mechanics when a Markov chain is needed that maximises the entropy under certain constraints. Note that for this proof we did not presume the existence of conditional expectations (rather we constructed them explicitly for finite σ -algebras); that is, we did not resort to the Radon-Nikodym theorem in a hidden way., n - 1, n and $En := \{e \in E : e \cap Bn = \emptyset\}$. Since M takes only integer values, there is a (random) n0 such that Mn = Mn0 for all $n \ge n0$. The graphs in Fig. We have bn1, p1 \le st bn2, p2 if and only if $(1 - p1)n1 \ge (1 - p2)n2$ (17.31) $n1 \le n2$. Then X and 1A are independent; hence E[E[X | F] 1A] = E[X | A]. The aim is to show that almost surely $B \in H\gamma$. Define fn : $[0, 1] \rightarrow R$ by fn (x) = 2n f d\lambda, if k = 2n f d\lambda, if k = 2n f d\lambda. is chosen such that $x \in Ik, n$. $(f + g) - d\mu - f d\mu + g d\mu - g - d\mu$ f $d\mu + g d\mu - g - d\mu$ f $d\mu + g d\mu$., N - 1. As in the calculation for X, we obtain (since H (ν) = 0) $1 \ge P[\xi n (Y) = \nu] = #An (\nu) \le nH(\nu)$; hence #An (ν) $\le nH(\nu) = 0$ and $\kappa t \le 0$ (E) and $\kappa t \le 0$ (\in C0 (E) for every $f \in$ C0 (E). (iii) $\kappa(x, \cdot) =$ Poix is a stochastic kernel from $[0, \infty)$ to N0 (note that $x \rightarrow$ Poix (A) is continuous and hence measurable for all $A \subset$ N0). In the general case, for N,
the upper bound 2 max{ki : i = 1, . \blacklozenge i=1 Takeaways Consider an orthonormal basis of the Hilbert space L2 ([0, 1]) and assign to each basis vector an i.i.d. standard normally distributed factor. Let $n \leftarrow n + 1$. Starcise 15.1.7 Let μ be a probability measure on R. 19.5). Show that for any $\epsilon > 0$, there is a compact set $C \subset A$, a closed set 4.3 Lebesgue Integral Versus Riemann Integral 107 D $\subset R \setminus A$ and a continuous map $\phi: R \rightarrow [0, 1]$ with $1C \leq \phi \leq 1R$ and such that $1A - \phi 1 < \epsilon$. Then the exchangeable σ -algebra is P-trivial; that is, $P[A] \in \{0, 1\}$ for all $A \in E$. Reff $(0 \leftrightarrow k) + \text{Reff}(k \leftrightarrow n)$ Note that this yields the ruin probability of the corresponding Markov chain X on $\{0, ., (12.4), 262, 12\}$ Backwards Martingales and Exchangeability What happens if we let $N \rightarrow \infty$? 2.3 Kolmogorov's 0-1 Law 69 Show that for k = (k1, .) We have not shown that almost surely B was not Hölder-12 -continuous at any $t \ge 0$ (however, see Remark 22.4). What is the distribution of S := Tn=1 Xn? The states at later times are defined inductively by Xn (i) = Xn-1 (i), if In = i, Xn-1 (In + Nn), if In = i. For any $k \in N$, choose be (k) > b(k) such that $\mu(a(k), be(k)] \le \mu(a(k), b(k)] \le \mu(a(k), b(k)) \le \mu$ dbn,p n,q 7.5 Supplement: Signed Measures In this section, we show the decomposition theorems for signed measures (Hahn, Jordan) and deliver an alternative proof for Lebesgue's decomposition theorems. Clearly, for any $n \in N$,) * P[#C p (0) = ∞] \leq P there is an $x \in C$ p (0) with $x\infty = n$. (ii) Let A be a ring. 95 4.1 Construction and Simple Properties . Show that there exists a b \in R such that X = b almost surely. (1.3) In order to show that DE is a λ -system for any E $\in \delta(E)$; hence $\Omega \in DE$. Then F0 is a P-trivial σ -algebra. Hence the claim follows immediately from Corollary 12.18. Is this random walk recurrent? Let E be the set of possible outcomes. In larger markets, equivalence holds only with a somewhat more flexible notion of arbitrage (see [30]). Proof Define fn by fn (k/n) := f (k/n) for k = 0, . n \rightarrow \infty (i) There is a sequence (Pn)n \in N 1 (R), there is a sequence (Pn)n \in N 1 (R), there is a sequence (Pn)n \in N 1 (R) in M1 (R) such that each n $\rightarrow \infty$ Pn has finite support and such that Pn \Rightarrow P Show that X is an F-martingale. (iii) $A = \{(a, b] : a, b \in R, a \le b\}$ is a semiring on $\Omega = R$ (but is not a ring). Hence $0 \le g \le 1$. Fix some $N \in N$ and define $xj := \inf x \in R : F(x) \ge j/N$, $j = 0, ., \beta, 0$ In this case, $F \beta$ has a local maximum at 0 and has global minima $m \pm .$ For $F \in V$, we define $F := \sup\{|F(f)| : f = 1\}$. Proof We use different arguments to show that the right-hand side of (23.18) is a lower and an upper bound for the left-hand side. 518 21 Brownian Motion Proof Define Nt := {Xt = Yt} for t \in I and N⁻ = t \in I Nt . , x(n)). i \in I Theorem 8.7 (Bayes' formula)) Let *I be a countable set and let (Bi)i \in I and N⁻ = t \in I Nt . , x(n)). space with Borel σ -algebra B(E). If ϕ is convex, then E[$\phi(X) -] < \infty$ and E[$\phi(X) \ge \phi(E[X])$. Since for each $n \in N$, the left hand side is a probability measure, we have $\alpha 1 = 1$ and $\mu pn - \pi T V \le C|\lambda 2|n$ (18.10) for a constant C (that does not depend on μ). Then (Xn) $n \in N$ fulfills the strong law of large numbers. A function f is called harmonic on G := E \ A if (p - I)f = 0 holds on G. Furthermore, for all x1, If ϕ is convex, then ϕ is continuous and hence measurable. (ii) $\nu 0 \mu$. E In particular, the family (κn) $n \in N$ is a Markov semigroup and the Radon-Nikodym Theorem 188 As κ is an isometry, κ in particular is injective. The use of general descriptive names, registered names, trademarks, service marks, etc., N - 1, are eigenvalues of modulus 1. As an example consider X := exp(Y), where Y ~ N0,1., 2n-1. + Xn,kn. 9.2 Martingales 221 Remark 9.33 Many statements about supermartingales hold mutatis mutandis for submartingales. n=1 The same computation without absolute values yields the remaining part of the claim. Hence, by Theorem 1.96, there exist sets A1, A2, . Instead of checking by a direct computation that this process (X, Y) is indeed a coupling with transition matrix p, - consider the construction of Markov chains from Theorem 17.17: Let (Rn (x) $: n \in N0$, $x \in E$) be independent random variables with distribution P[Rn(x1) = x2] = p(x1, x2), and let $R^{\tau}n((x1, y1)) = (Rn(x1), Rn(y1))$. A necessary and sufficient condition for this to be true is that μ vanishes on the sets where ν vanishes. Hence $\tau := (\tau \wedge t) + 1\{\tau > t\}$ and $\sigma := (\sigma \wedge t) + 1\{\sigma > t\}$ (and thus $\tau + \sigma$) are Ft -measurable. (iii) If in addition $\mu(\Omega) = 1$, then (Ω, A, μ) is called a probability space. We will show that Pp [N = m] = 0 for any $m \in N \setminus \{1\}$. This is k-1 equivalent to the condition that $m1 \le m2 \le .$ From these considerations and from Theorem 16.12, we conclude the following theorem. Hence (PXn) n \in N + Yn is integrable and E[Xn Fm] = Xm if m < n (where $F = \sigma$ (X)). n Proof Let X be the canonical process on (Ω , A, P) = (Rd)N, B(Rd) \otimes N, P and let $\tau : \Omega \to \Omega$ be the shift; that is, $Xn = X0 \circ \tau n$. Evidently, \leq st is a partial order on M1 (Rd). Bn is a disjoint union of certain sets Cn, 1, . By Corollary 15.3, C is a separating class for Mf ([0, ∞]) and thus also for Mf ($[0, \infty)$). (21.45) 0 Lemma 21.46 (κt) $t \ge 0$ is a Markov semigroup and there exists a Markov process (Yt) $t \ge 0$ with transition kernels Px [Yt \in dy] = κt (x, dy)., xn) $\rightarrow xi$ be the projection on the ith coordinate for each i = 1, . Hint: Use Exercise 4.2.4 to show the assertion first for indicator functions, then for simple functions and finally for general $f \in Lp(\lambda)$. Let X0 = 0 almost surely and P[X1 = -1] = P[X1 = 0] =Semigroups 323 Theorem 14.45 (Kernel via a consistent family of kernels) Let I \subset [0, ∞) with 0 \in I and let (ks,t : s, t \in I, s < t) be a consistent family of stochastic kernels on the Polish space E. Exercise 4.3.1 Use Theorem 4.26 to compute 31 0 log(x) dx and 3π 0 sin(x) dx. Clearly, A remains unchanged if we change p the state of finitely many edges. In >0 a>0 n \in N p \in [0, 1] r > 0 p \in (0, 1] $\lambda>0$ uniform U[-a,a] triangle Tria N.N. Gamma $\Gamma\theta$, r exponential exp θ two-sided exponential exp(\theta) two-sided exponential exp(\theta) two-side $x \theta = -\theta x \frac{1}{2a} + 1 = 1 - \frac{|x|}{a} [0, a] = 0$ uniform U[0,a] [-a, a] $1/a R \mu \in R \sigma^2 > 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weights $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma$ Density / Weight $2\sqrt{1} \exp(-(x-\mu)^2 2\sigma^2) = 0$ normal Nµ, $\sigma^2 2\pi\sigma^2 2\sigma^2$ normal Nµ, $\sigma^2 2\sigma^2 2\sigma^2$ normal Nµ,194 8 Conditional Expectations Definition 8.9 Let $X \in L1$ (P) and $A \in A$. Then (owing to (P2)) *) *) $\alpha(s + t) = E N(0,s] + E N$ probability. 2 Hence (fZ, $Z \in Z$) is uniformly integrable by Theorem 6.24(iii). If $k \in N$ and $\sigma k < \infty$,
then clearly Yoi $-Y\tau i = Y\sigma i - a \ge b - a$ for all $i \le k$; hence (H · Y) $\sigma k = \sigma i k$ (Yj $-Y\tau i = Y\sigma i - a \ge b - a$ for all $i \le k$; hence (H · Y) $\sigma k = \sigma i k$ (Yj $-Y\tau i = Y\sigma i - a \ge b - a$ for all $i \le k$; hence (H · Y) $\sigma k = \sigma i k$ (Yj $-Y\tau i = Y\sigma i - a \ge b - a$ for all $i \le k$; hence (H · Y) $\sigma k = \sigma i k$ (Yj $-Y\tau i = Y\sigma i - a \ge b - a$ for measurable A $\subset E$, the intensity measure yields the expected value of this random variable. For $x \in (x_j - 1, x_j)$, we have (by definition of xj) Fn (x) \leq Fn (xj -) \leq F (xj -) + Rn \leq F (xj -1) - Rn \geq F (xj -1) - Rn \geq Fn (xj -1) - Rn \geq F (xj -1) - Rn \geq Fn (xj -1) - Rn = Fn (xj -1) - Rn \geq Fn (xj -1) - Rn = Fn (xj -1) - Rn \geq Fn (xj -1) - Rn = Fn (xj -, BtN)]. , mr \in N0 , we also have ri=1 r ki mi \in N(x, x). Let N = ∞ n=1 B (1/n). *) Dominated convergence yields lim sup E f (Xn) – f (Yn) = 0. Recall that a function g : Rn \rightarrow R is called affine linear if there is an a \in Rn and 7.2 Inequalities and the Fischer-Riesz Theorem 169 a b \in R such that g(x) =)a, x* + b for all x. In particular, { $\tau \land t \leq s$ } \in Fs \subset Ft for any s \leq t., C(tn), 2f $\sim -\gamma$. If y \in E with p(x, y) > 0, then Lx, y = 1 and hence Lx0, y = Lx0, x + Lx, y = i + 1 (mod d)., Yn). Let (a, b], (a(1), b(1)], (a(2), b(2)], . \Rightarrow Chapter 8 Conditional Expectations If there is partial information on the outcome of a random experiment, the probabilities for the possible events may change. Indeed, let $\Omega = \{0, 1\}$ N. Let (a, b], (a(1), b(1)], (a(2), b(2)], . and let P be the product measure $\otimes N$ (Theorem 1.64), as well as $A = \sigma ([\omega 1, .(2) With probability \pi(x I, \sigma)/\pi(x-I), replace x by x I, \sigma . 19.7 Initial situation. Clearly, E[NI] = <math>\alpha (I) < \infty$; thus we have (P4). $k=1 \mu$ This implies that $\mu^{\sim} F$ is σ -subadditive. However, if we assume the existence of higher moments, we get reasonable estimates on the rate of convergence. t $\rightarrow 0$ If I = [0, ∞) and if in addition $\nu t \rightarrow \delta 0$, then the convolution semigroup is called continuous (in the sense of weak convergence). Hence, let c := $-E[\log(0)] > 0$. Hence, Z is a martingale, and the first six centered moments are Ei [(Zn - i)2] = 2in, Ei[(Zn - i)3] = 6in2, Ei[(Zn - i)4] = 24in3 + 12i2n2 + 2in, Ei[(Zn - i)5] = 120in4 + 120i2n3 + 30in2, Ei[(Zn - i)6] = 720in5 + 1080i2n4 + (120i3 + 360i)n3 + 60i2n4 + (120i3 + 360i)n3 + (120i3 +multiplication by a and $b^{\sim} := ab + d + a$ (1{|x| 0, m, n \in N. If $\mu = 0$, then f d $\mu = 0$ for all f; hence $\nu(\Omega) = 0$ and thus $\nu 0 \mu$. = Yn = 1 X = x]? 303 304 307 317 322 15 Characteristic Functions and the Central Limit Theorem ... Consider the first time that X is in K: $\tau K := inf\{t \in I : Xt \in K\}$. $\in L1$ (P) be pairwise independent and identically distributed Later, Ω will be interpreted as the space of elementary events and A will be the system of observable events. Hence, by (i), for any $\epsilon > 0$, $(1 - \epsilon) \alpha i \mu(Ai) = (1 - \epsilon) \alpha i \mu(Ai) = (1 - \epsilon) \alpha i \mu(Ai) = (1 - \epsilon) \alpha i \mu(Ai)$ = $(Xt, t \in I)$ (on (Ω, F, P)) with values in (E, E) is called a stochastic process with index set (or time set) I and range E., xn., λN ..); hence $n \to \infty$ PII [X $\in A$ X0, . Then we construct a version of X that has continuous paths, the so-called Wiener process or Brownian motion. Hence the map En $\rightarrow [0, 1]$, $\nu \to PI[\xin(Y) = \nu]$ is maximal at $\nu = \nu$. For $n \in A$ X0, . Then we construct a version of X that has continuous paths, the so-called Wiener process or Brownian motion. N, define the class of cylinder sets that depend only on the first n coordinates An := { $[\omega 1, . By properties (P2) and (P3), the random variables (Xn (k), k = 1, . \bullet Theorem 1.18 (<math>\cap$ -closed λ -system). Let D $\subset 2\Omega$ be a λ -system. Theorem 2.7 (Borel-Cantelli lemma) Let A1, A2, . However, in general, $\tau - s$ is not. Hence the waiting time for the next click is exponentially distributed with parameter α . A As functions of B, both sides are finite measures on B(R) that coincide on the \cap stable generator ($-\infty$, r], r \in Q. 18.1 Periodicity of Markov Chains 1/2 3 1 437 2 1/2 5 1 6 1 1 1 4 1 1 8 1 7 Fig. This property is called infinite divisibility and is shared by other probability distributions such as the Poisson For continuous $f:[0, \infty) \rightarrow [0, \infty)$, let $\infty \phi f(z) = t z - 1 f(t) dt 0$ for those $z \in C$ for which the integral is well-defined. Assume $Xn \rightarrow X$ and $d(Xn, Yn) \rightarrow 0$ in D probability. This conclusion from the convergence theorem for backwards martingales will be used in an essential way in the next section. By the induction hypothesis, E[X2k] = (-1)k u(2k)(0) for all k = 1, . Replacing g by $g^{\tilde{}} := |g| \operatorname{sign}(f)$ (note that $g^{\tilde{}} p = gp$), we obtain $\kappa(f) p \operatorname{gp} \ge f g^{\tilde{}} d\mu = fg1$. For any $p \in [0, 1]$, define $\otimes n \operatorname{Pp} = (Berp) \otimes n = (1 - p)\delta 0 + p\delta 1$. "(i) $\Rightarrow (ii)$ " By Theorem 7.7(iii), for any $x 0 \in I$, the map $x \to \phi(x0) + (x - x0) D + \phi(x0)$ is in $L(\phi)$. Furthermore, show that such a process can be constructed in such a way that almost surely the map t \rightarrow Xt is monotone increasing and right continuous. Now let $l \in \{1, . Then there would be an \epsilon > 0 and an A \in A with \mu(A) > 0$ such that $\epsilon\mu(E) \leq \nu_s(E)$ for all $E \subset A$, $E \in A$. By Example 7.39, this implies that X is uniformly integrable. i=1 Now $\left[|Xi-1| + Ri, |i| \leq |Xi-1| - Ri, |i| \leq 1, if Xi-1 > 0, if Xi-1 < 0, if$ = 0. Applying Lévy's continuity theorem to Example 15.16, we get a theorem of Pólya. One way to access an online phone book is through the browser of your mobile device. 2.3 Binary tree. Note that $\log(1 + x) \le x$ for x > -1 with equality if and only if x = 0. Hence $D - \phi(x)$ and $D + \phi(x)$ are the minimal and maximal slopes of a tangent at x. ∞ (iv) Let x1, x2, The class of Lipschitz-continuous functions is a separating family for finite measures and for Radon measures. Consider the measures μ and ν that are restricted to F. Hence $3 \text{ S}^{\sim} dP = \infty \text{ n} = 1 (1 - 2 \text{ n} - 1) p(1 - p)n - 1 = -\infty \text{ since } p \le 12$. Then: *) (i) Var[X] = E (X - E[X]) $2 \ge 0$. (5.20) Write n 1(ti-1,ti] (X1). By the dominated convergence theorem (Corollary 6.26), we get $F(xn) = n \rightarrow \infty$ fn $d\mu \rightarrow f(\cdot, x0) d\mu = F(x0)$. As μ is (finitely) subadditive (see Lemma 1.31(iii)), we obtain $0 \in \epsilon + \mu((a(k), b(k)]) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + k = 1 \infty \mu((a(k), b(k))) \leq \epsilon + \mu((a(k$ have $(X_j) - 1$ $(A_j) \in A_j$ for all $A_j \in A_j$, and hence $A_j \subset A_j$. Then *) P Sn = 0 for infinitely many $n \in N = 1$. $|x|^3$ Then $E[X_{12}] = \infty$ but there are numbers A1, A2, . Clearly, x = (1, 0, .., 350 + 15) Characteristic Functions and the Central Limit Theorem Lemma 15.31 For $t \in R$ and $n \in N$, we have n-1 n it e-1 - it - . For every $x \in Zd$, we define the (random) open cluster of x; that is, the connected component of x in the graph (Zd, Ep): Cp(x) := { $y \in Zd : x \leftrightarrow p y$ }. \blacklozenge Proof This is left as an exercise. Hence X is stationary if and only if (Ω , A, P, τ) is a measure-preserving dynamical system. Define m = m(ν) := $\nu(\{1\}) - \nu(\{-1\})$. (iii) The set of all probability measures on a measurable space is a convex set. P j \in J k \in K j \in J k \in K j \in J k \in K i an independent family of real random variables, then also (Yn)n \in N is independent family of real random variables, then also (Yn)n \in N is independent family of real random variables, then also (Yn)n \in N is independent. For example, we study percolation theory at the point where we barely have measures, random variables, and independence; not even the integral is needed. Starting from a normed vector space X (here X = Cb (E) with the norm $\cdot \infty$), we define the $n \rightarrow \infty$ weak* -topology on the dual space X by writing $\mu - \rightarrow \mu$ if and only if $n \rightarrow \infty$ each μ defines a continuous linear μn (x) $- \rightarrow \mu(x)$ for all $x \in X$. If X has period $d \ge 2$, and if $n \in N$ is not a multiple of d, then, by Theorem 17.52, ; ; $\delta x pn - \pi$; ≥ 0 $|pn(x, x) - n({x})| = n({x}) > 0$. ('Hence, for A = 0, 14, we have $A \cap \tau - kn(A) = \emptyset$. Proof Since $E \subset \sigma(X1, X2) = n({x}) > 0$. ('Hence, for A = 0, 14, we have $A \cap \tau - kn(A) = \emptyset$. In order to show that $\mu *$ is an outer measure, it only remains to check that $\mu
* is \sigma$ -subadditive. Takeaways A reversible Markov chain is stationary and fulfills. an even stronger equilibrium condition: The condition of detailed balance says that on average the Markov chain jumps from x to y as often as it jumps from y to x (for all x and y). Proof Let $\alpha := \sup \phi(A) : A \in A$. A parameter of this distribution is the inverse temperature $\beta = T1 \ge 0$ (with T the absolute temperature). In the latter case, we also get convergence of the Markov chain to the invariant distribution). Thus, for every $\epsilon > 0$, there exists a compact set A \subset C([0, ∞)) with Pi (A) > 1 - ϵ for every i \in I. Secondly, the network can be reduced in a series of elementary steps: Resolving parallel connections, resolving serial connections and resolving intermediate points using the star-triangle transformation. If f1 and f2 are densities of ν with respect to μ , then f1 = f2 μ -almost everywhere. (18.8) In particular, π is invariant (check this!). We construct a filtration F = (Fn) $n \in \mathbb{N}$ by letting Fn := σ ({A1 , . A map f : [0, ∞) \rightarrow E is called RCLL (right continuous with left limits) or càdlàg (continue à droit, limites à gauche) if f (t) = f $(t+) := \limsup t f(s)$ for every $t \ge 0$ and if, for every $t \ge 0$ and if inite. k=0 By Lemma 20.15, (Yn)n $\in \mathbb{N}$ is uniformly integrable, and by Birkhoff's ergodic $n \rightarrow \infty$ theorem, we have Yn $\rightarrow 0$ almost surely. 426 17 Markov Chains Consider now the signed measure $\mu = \pi - \nu$. 18.2. For $x \in E$, define X0x = x and x $X_{nx} = Rn (X_n - 1)$ for $n \in N$. The moduli of the sequence (xn) $\lambda | = f(2\pi k/N)$, where $f(\theta) = 21 - 4r(1 - r) \sin(\theta) 2$ for $\theta \in R$. Then μ is a Lebesgue-Stieltjes measure if and only if the sequence (xn) $\lambda | = f(2\pi k/N)$, where $f(\theta) = 21 - 4r(1 - r) \sin(\theta) 2$ for $\theta \in R$. Then μ is a Lebesgue-Stieltjes measure if and only if the sequence (xn) $\lambda | = f(2\pi k/N)$, where $f(\theta) = 21 - 4r(1 - r) \sin(\theta) 2$ for $\theta \in R$. Then μ is a Lebesgue-Stieltjes measure if and only if the sequence (xn) $\lambda | = f(2\pi k/N)$, where $f(\theta) = 21 - 4r(1 - r) \sin(\theta) 2$ for $\theta \in R$. Then μ is a Lebesgue-Stieltjes measure if and only if the sequence (xn) $\lambda | = f(2\pi k/N)$, where $f(\theta) = 21 - 4r(1 - r) \sin(\theta) 2$ for $\theta \in R$. Then μ is a Lebesgue-Stieltjes measure if $\lambda = 1$. constant. On the other, the strong law of large numbers claims $n \rightarrow \infty$ that for fixed ω , we have Sn (ω)/n $- \rightarrow 3.5$ (Fig. Therefore, Xt = X Reflection Why cannot we drop the assumption that $t \rightarrow E[Xt]$ be right continuous. (ii) If p0 + p1 = 1, then all of the statements are obvious. 2 Then X is a martingale; however, lim supn $\rightarrow \infty$ $Xn = \infty$ and lim infn $\rightarrow \infty Xn = -\infty$. Note that $\tau x1 > 0$ even if we start the chain at X0 = x. Assume that F1, F2, ..., Xtk). In other words, if Z, Z 1, . \diamond 7 Lp -Spaces and the Radon-Nikodym Theorem 170 Example 7.13 Let $G = [0, \infty) \times [0, \infty)$, $\alpha \in (0, 1)$ and $\phi(x, y) = x \alpha y 1 - \alpha$. X is called (with respect to F) a martingale if E[Xt Fs] = Xs for all s, t \in I with t > s, submartingale if $E[Xt Fs] \ge Xs$ for all s, $t \in I$ with t > s, supermartingale if $E[Xt Fs] \le Xs$ for all s, $t \in I$ with t > s. In other words, we have ft $d\nu n \rightarrow ft$ (0) = -t 2/2. Later we will encounter a different (but equivalent) definition that will, however, rely on the notion of an integral that is not yet available to us at this point (see Definition 14.20). The general solution is unknown; however, for the case r = 2, Sylvester [163] showed that N = (k1 / dx - 1)(k2 / dx - 1) is minimal. 3 6 2 Thus, for $|t - s| < \delta$ by Lemma 15.20, $|\phi\mu(t) - \phi\mu(s)| \le \epsilon$. i=1 j =1 Theorem 5.8 (Cauchy-Schwarz inequality) If X, $Y \in L^2(P)$, then Cov[X, Y] $2 \le Var[X]$ Var[Y]. j =1 i=1 Proof If $\mu(Ai \cap Bj) > 0$ for some i and j, then $Ai \cap Bj = \emptyset$, and $f(\omega) = \alpha i = \beta j$ for any $\omega \in Ai \cap Bj$. are E1 -valued random variables with $P[X \in U\phi] = 0$ and $D \ Xn \to X$, then $\phi(Xn) \to \phi(X)$. Theorem 15.2 (Stone-Weierstraß) Let E be a compact Hausdorff space. 116 5 Moments and Laws of Large Numbers $n \propto Proof$ Define Sn = i=1 Xi for $n \in N0$. 9.1 Processes, Filtrations, Stopping Times We introduce the fundamental technical terms for the investigation of stochastic processes (including martingales). Thus we define $R := R \cup \{-\infty, +\infty\}$. & Exercise 12.1.2 Derive equation (12.4) formally...) $\in E N$, let $\xi_n(x) := n1$ nl=1 δx l be the empirical distribution of x1,.., d - 1) be another decomposition that satisUniqueness Let (E $0 = \emptyset$ (otherwise fies (18.5) and (18.6). Hence $\phi exp2$ (t) = $\phi exp\theta$ (t) $\phi exp\theta$ (-t) = $\theta 1 1 1$. By Ohm's rule, we get u(1) = u(0) + I (x1) R(0, 1), u(2) = u(1) + I (x1) R(1, 2) x0 = 1 C(0, 1) u(0) = 0 C(1, 2) x1 = 6 C(5, 6) u(6) = 1 Fig. It was a single printed card with some numbers printed on it. + Thr $n \in N$ and let $S := \{T1s + . By \text{ the Radon-Nikodym theorem (Corollary 7.34)}$ (applied to the measurable function f with $\nu = f \mu$. For measurable maps $f : E n \rightarrow E$ and $F : E N \rightarrow E$, define the maps f and F by f (x) = f (x) and F (x) = F (x). Now let z := x - y. Consider now the situation that we studied with the summation formula for conditional probabilities. A We want to extend the real line by the points $-\infty$ and $+\infty$. Hint: Use Markov's inequality for f (x) = ey x and choose the optimal γ . Then lim sup $\mu(F) \leq \inf \lim_{n \to \infty} \rho F_{,\varepsilon} d\mu = \mu(F) \epsilon = 0$ since $\rho F_{,\varepsilon} (x) \to 1F$ (x) for all $x \in E$. + Hence the integral is an extension of the map I from E to the set of nonnegative measurable functions. Now apply Corollary 15.33 to the random variable X with respect to the probability measure $P[\cdot | Y \in A]$. To this end, consider the map $\varphi n : [0, 1) \rightarrow \{0, 1\}n \ x \rightarrow 1[1/2, 1)(x)$, 1[1/2, 1)(x), 1[1/ $\nu(A \cap E)$. Use a suitable orthonormal basis on [0, 1]d to show: (i) There is a Gaussian process (Wt)t \in [0,1]d with covariance function Cov[Wt , Ws] = d ti \wedge si . The outcomes of the other experiments do not play a role for the occurrence of this event. (T owerproperty) E[E[X | G] | F] = E[X | G]. 1 We will show that P[B] = 1 if and only if ∞ n=0 wn < ∞ . Proof Clearly, X-1 (E) \subset X-1 σ (E) = σ X-1 (σ (E)). Output F (*). We will obtain 1 - F (x1, x1) as the limit of the probability that a random walk started at x1 hits A0 before returning to x1 as A0 $\downarrow \emptyset$. n- ∞ Definition 13.17 Let X, X1, X2, . For x = (x1, x2, .13.3 Prohorov's Theorem 293 The other implication in Prohorov's Theorem 293 theorem is more difficult to prove, especially in the case of a general metric space. Since all eigenvalues are different, every eig degree of $h \in HL$; that is, the number of neighbors of h in HL. 616 24 The Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ 142 5 Moments and Laws of Large Numbers Using the Poisson Point Process Exercise 24.1.1Let X1, X2, . $n \rightarrow \infty$ $n \rightarrow \infty$ nproduct measure if e p(e) \in (0,
∞) \ {1}? Using the strong Markov property, we get that, for all $z \in N0$, P0 [B] = 1. Consider the class of sets A := V \cap C : V \subset E open, C \subset E closed. Theorem 5.36 The family (Nt, t ≥ 0) is a Poisson process with intensity α . Corollary 1.87 With the above notation, $\tau^{-} = \tau$ and hence B R = B(R). n=1 Example 12.15 Let X1, X2, . & Exercise 11.2.2 X1, X2, . For example, the assumptions are introduced such as bounded variances. We infer Pp [NL1 ≤ NL0 - 1] ≥ Pp [A2L,0] > 0, which leads to a contradiction. Let $\rho K_{,\epsilon}$ be the map from Lemma 13.10. The CFP of CPoiv is given by ϕv (t) = exp (eit x - 1) v(dx). Theorem 17.14 If $I \subset [0, \infty)$ is countable and closed under addition, then every Markov process (Xn) neI with distributions) For some distributions P with density $x \rightarrow f(x)$ on R or weights P ({k}), $k \in N0$, we state the characteristic function $\phi(t)$ explicitly: [-a, a] R {0, . + xk}. N}, and the transition matrix is of the form $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $t \in [0, 1]$, $t \in$ with $\sqrt{\text{respect to W}}$. k=1 520 21 Brownian Motion Now fix $\gamma \in (0, \beta/\alpha)$ to obtain P[Bn] $\leq \infty$ P[Am] $\leq C$ m=n 2-($\beta-\alpha\gamma$) n n $\rightarrow\infty$ $\rightarrow 0, 1 - 2\alpha\gamma - \beta$ (21.6) hence P[N] = 0., n} geometric distribution HypB,W ;n on $\{0, . It is reasonable to assume that a market gives no opportunity for an arbitrage. Denote by B + (E) the set of measurable maps E \rightarrow [0, \infty] and by BbR (E) the set of measurable maps f \in B + (E), 3 E \rightarrow R with compact support., n : Xk = i \}$ for i = 1, . Hence (An (ϕ))n >k is a backwards martingale with respect to (E-n)n \in -N . Theorem 8.29 (Regular conditional distributions in R) Let Y : $(\Omega, A) \rightarrow R$, B(R) be real-valued. Therefore, L[X $[\Xi \infty] = \Xi \infty$ Remark 12.27 (i) In the case considered in the previous theorem, by the strong law of large numbers, for any bounded continuous function f : E \rightarrow R, $n \rightarrow \infty$ f d $\Xi n \rightarrow \infty$ f d Ξ Aj F almost surely. By (11.1), we have $E[X \propto 1B] = Q(B)$ for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$ Fn and thus also for all $B \in r \in N$. Reff $(0 \leftrightarrow x) = \text{Reff}(1 \leftrightarrow x) = \text{Reff}(1 \leftrightarrow x) = \text{Reff}(0 \leftrightarrow 1) = 32 + 2726 + 2726 + 2726 + 2726 + 2726 + 2726 + 2732 = 17$, 24 = 5, 6 = 29. 24.3 The Poisson-Dirichlet Distribution. Proof We show equality by showing the two inequalities separately. Since $\Lambda(0) = 0$, we have $\Lambda * (x) \ge 0$ for every $x \in \mathbb{R}$. However, if the σ -algebra A can be approximated by countably many A sufficiently well, then there is hope. That is, (Y1, . On the other hand, if X has the Markov property, then (see (8.6)) P[Xt = i Xsn = in] for almost all $\omega \in \{Xsn = in\}$. These sets are not easy to construct like, for example, Vitali sets that can be found in calculus books (see also [37, Theorem 3.4.4]). In the next theorem, we will see that this is still true even if ϕ is not twice continuously differentiable when we pass to the right-sided derivative D + ϕ (or to the left-sided derivative), which we show always exists. Then, for B \in A, we would have $(f + \varepsilon 1A) d\mu = f d\mu + \varepsilon \mu(A \cap B) B B \leq \nu a (B) + \nu s (B) = \nu(B)$. (2n)! The two processes (Xn)n \in N0 and (Yt)t \in [0, ∞) thus have the same Green function. 30 1 Basic Measure Theory Let A, A1, A2, are i.i.d. and \sim Rad1/2 : k=1 P[R1 = 1] = P[R1 = -1] = 1 . Proof The two statements arec immediate consequences of de Morgan's rule Ai). N - 1} the Markov chain with transition matrix p(x, y) = 1{y=x+1(mod N)} . The ringshaped superconductors have melted down to single points. However, $G(\infty) = 0 < \lim \text{supn} \rightarrow \infty$ Gn (∞) = 1; hence we do not have weak convergence here either. Hence, let f : RN \rightarrow R be continuous and bounded and F (B) = f (Bt1, . $\in [0, 1]$ are such that ∞ pk = 1. However, it has the advantage that it can be performed without going through all the network reduction steps if, for some reason, we know the effective resistances already. ♦ Definition 5.22 (Empirical distribution function) Let X1 , X2 , . Hence, it only remains to show that I[~] = A* . Then τn is the time of the next click? " = A* . Then τn is the ti " Let X be aperiodic. Now assume that (17.33) holds for some given t \in N0. Consequently, for such a random walk, one out of three alternatives holds: (i) The random walk goes to ∞ at positive speed. Formally, we call three events A1, A2 and A3 (stochastically) independent if P[Ai \cap Aj] = P[Ai] \cdot P[Aj] for all i, j \in {1, 2, 3}, i = j, (2.2) and P[A1 \cap A2 \cap A3] = P[A1] \cdot P[A2] \cdot P[A3]., 2n. 76 2 Independence Such a family (Ye) e \in exists by Theorem 2.19. m $\rightarrow \infty$ m $\rightarrow \infty$ However, by definition, $\alpha \ge \phi(\Omega +)$; hence $\alpha = \phi(\Omega +)$. To this end, we need a representation of convex functions of many variables as a supremum of affine linear functions. = F{1,...,n} (x, Exercise 14.1.1 Show that Ai = i \in I ZJ. Then, for any $\varepsilon > 0$, |Sn| 1/2 $(\log(n))(1/2) + \varepsilon n \rightarrow \infty n$ lim sup = 0 almost surely. 1 Exercise 4.2.1 Let (Ω, A, μ) be a measure space and that 3 let $f \in L_3(\mu)$. Show that $\mu \in M$ is extremal if and only if τ is ergodic with respect to μ . ; ; 3 3 For $f \in L_2(\mu)$, define ; f = f p for any $f \in f$. We construct the Brownian bridge as follows. This approach allows for an independent proof of de Finetti's theorem. (iii) Any countable (respectively finite) union of sets in A can be expressed as a countable (respectively finite) disjoint union of sets in A. If the test gives an alarm, what is the probability that the device just tested is indeed defective? n < N n 512 20 Ergodic Theory Definition 20.34 (Kolmogorov-Sinai entropy) The entropy of a (general) measure-preserving dynamical system (Ω , A, P, τ) is h(P, τ ; P), P where the supremum is taken over all finite measurable partitions of Ω . However, now H (x i) – H (x) = 1{x(j) = x(i)} j : j ~i = -2 1{x(j) = x(i)} j : j ~i = -2 1{x(j) = x(i)} j : j ~i = -2 1 variables., Bτ n +tN) Fτ n) * = EBτ n f (Bt1, . The procedure we used here to derive Pólya's theorem has the disadvantage that it relies on the local central limit theorem, which we have not proved (and will not). (ii) Let X be exponentially distributed with parameter θ > 0. Reflection In Theorem 1.53, in general, μ cannot be extended to a measure on all of 2Ω . " \leftarrow " Now assume that α and ν are given. be i.i.d. random variables with values in Σ and with distribution PX1 = μ . Hence also the countable union of these sets is in Ft: { $\tau K \leq t$ } = {Xs $\in K$ } \in Ft. The above requirements translate to: (NI, I \in I) being a family of random variables with values in Σ and with distribution PX1 = μ . Hence also the countable union of these sets is in Ft: { $\tau
K \leq t$ } = {Xs $\in K$ } NI \cup J = NI + NJ if I \cap J = \emptyset and I \cup J \in I. 5.) Let A3L:= x 1, x 2, x 3 \in BL |BL-1i=j| 3 {C p (x i) \cap C p (x j) = \emptyset } \cap {#C p (x j) \cap C p (x j) = 0 \cap {#C p (x j) \cap C p (x j) = 0 \cap {#C p (x j) \cap C p (x j) \cap C p (x j) = 0 \cap {#C p (x j) \cap C p (x j) Finetti) The family $X = (Xn) \in N$ is exchangeable if and only if there exists a σ -algebra $A \subset F$ such that (Xn) $\in N$ is i.i.d. given A. Let $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$, and let Ai be the event where $A^{\tilde{}}$ i $\subset E$ for any 1 n $\in E$ e i=1 i $\in N$. 2-n 2n Y!. However, An (ϕ) is En -measurable and hence || *) 1 1 $\phi(X)$ En $] = \phi(X)$. We define the convolution of μ and ν as $\mu * \nu = PX+Y$. By (21.5) and (21.11), we Hence)X * is a modification of X. 18.2 Coupling and Convergence Theorem 443 Corollary 18.10 If X is an irreducible random walk on Zd, then every bounded harmonic function is constant. Defining $c + = \alpha \nu([1, \infty))$ and $c - := \alpha \nu((-\infty, -1])$, we get (16.22) and thus (iii) and (i). Assume that initially all edges have resistance 1. (1.12) k=1 For this purpose we use a compactness argument to reduce (1.12) to finite additivity. Then $cX \in L1$ (P) and $X + Y \in L1$ (P) as well as E[cX] = c E[X] and E[X + Y] = E[X] + E[Y]. "(iii) \Rightarrow (ii)" Assume (iii) and $\mu(\Omega) < \infty$. Theorem 3.11 (Extinction probability of the Galton-Watson process) Assume p1 = 1. Jensen's inequality (Theorem 4.26 Let $f : \Omega \to R$ be measurable and $f \ge 0$ almost everywhere. For Markov chains, the entropy can be computed explicitly. Finally, for $x \in E$ let SAn (x) = $y \in E$: (pA)n (x, y) > 0, for $n \in N0$ and SA (x) = ∞ SAn (x) = $y \in E$: FA (x, y) > 0. The distribution of (IA1, . Hence, let $A \subset E$ be open. Hence now consider the event $B := \{E[X | F] | is an interior point of I \}$. The sets $A \in \tau$ are called open, and the sets $A \subset \Omega$ with $Ac \in \tau$ are called closed. The following theorem however, shows that the measure of a set from σ (A) can be well approximated by finite and countable operations with sets from A. n=0 Corollary 17.49 If X is positive recurrent, then π := distribution for any x \in E. 19.9) if the resistances R1 , R2 , R3 , R Ri R[~] i = δ for any i = 1, 2, 3, (19.18) where δ = R1 R2 R3 R1-1 + R2-1 + R3-1 = 2 R 3 1 R R. Without loss of generality, assume μ (An) < ∞ and hence U(An) = \emptyset for all $n \in N$. Since ϕ (0) < 0, we have $\tau > 0$. Then $h \equiv 1$ serves the purpose. By virtue of Lévy's continuity theorem, one can show that (see Exercise 16.1.2) $\phi(t) = 0$ for all $t \in R$ if ϕ is infinitely divisible. n Together with Theorem 12.17, we conclude that *) $n \rightarrow \infty$ An ($\phi k - 1$) An (fk) $- \rightarrow E \phi k$ (X1, . *** With a lemma, we prepare for an alternative proof of Lebesgue's decomposition theorem (Theorem 7.33). ∞ Define X := n=1 $\lambda n Xn$. In all, then, for θ , we get N - 1 different values (note that the complex conjugates of the values considered here lead to the same values λn), $\theta n = e(n/N)\pi i$ for n = 1, . * Exercise 18.2.4 Let X be a Stochastic Ordering and Coupling 433 Assume that (17.32) hold. In order for this to hold for the righthand side, by Weierstraß's theorem on rearrangements of series, the series has to converge absolutely. Use Exercise 15.4.1 to show that the random variables X := BZ and Y := (1 - B)Z are independent with $X \sim \Gamma_1$, r and $Y \sim \Gamma_1$, s. Hence let $m \in N$ and 0 = t0 < t1 < . Proof Let a > 0 and $n \in N$. A circle is a self-avoiding (finite) path that ends at its starting point. Then the usual class of open sets is the topology $\tau = (x, r) \in F$ Br $(x) : F \subset \Omega \times (0, \infty)$. and $X^{\sim} 1$, $X^{\sim} 2$, . , tn }, we have $\kappa(x, \cdot) \circ XJ - 1$ $n - 1 = \delta x \otimes \kappa tk + 1$ -tk. and This implies (i). Definition 19.1 Let $A \subset E$..) for $n \in N$. As shown in the first part of this proof, almost sure convergence of ∞ n=1 Xn and (i) imply that ∞ n=1 Yn converges almost surely. (16.9) k=1 (In ∞ contrast to the situation in Theorem 3.3 (Multiplicativity of generating functions) If X1, Example 7.12 Let X be a real random variable with $E[X2] < \infty$, I = R and $\phi(x) = x 2$. In fact, X is stationary if only Y is stationary. Theorem 15.51 (Kolmogorov's three-series theorem) Let X1, X2, This is a microscopic model for a magnet that assumes that each of n indistinguishable magnetic particles has one of two possible orientations $\sigma \in \Sigma = \{-1, +1\}$. Then X is a Brownian motion (17.2) 394 17 Markov Chains Conversely, for every Markov process X, Equation (17.2) defines a semigroup of stochastic kernels. Define Xi, j := E[Xi F]. With the Poisson distribution, we have encountered such a limit distribution that occurs as the number of very rare events when the number of possibilities goes to infinity (see Theorem 3.7). In the figures we label each edge with its resistance if it differs (in the course of the reduction) from 1. \bullet Example 2.25 Let $\mu i \in \mathbb{R}$ and $\sigma i 2 > 0$ for $i \in I$. We will need a theorem on conservation of energy and Thomson's principle) on the minimization of the energy dissipation. Evidently, $\mu(\emptyset) = 0$, and it is simple to check that μ^{\sim} is σ -finite. Let $C := (C(x, y), x, y \in E)$ be a family of weights with C(x, y) = 0 for all $x, y \in E$ and $C(x) := C(x, y) < \infty$ for all $x \in E$. be ranD $n \to \infty$ dom variables with values in E. By 2 ak x + T heorem 15.13, μk has the characteristic function $\phi \mu k$ (t) = 1 - |tak|. Definition 2.20 For any $i \in I$, let Xi be a real random variable. 4 36 1 Basic Measure Theory 1.4 Measurable Maps A major task of mathematics is to study homomorphisms between objects; that is, structure-preserving maps. Choose a subsequence (fnk) $\geq \mu^{n}$ n+1 (W n), but equality does not hold in general. Of course, for any n, the actual value of Sn will sometimes be smaller than n E[X1] and sometimes larger. Clearly, p is irreducible, and p is aperiodic if and only if N is odd. In either case, where does the proof of Theorem 5.34 fail? To this end, we have to modify the rescaled processes so that they become continuous. $n \rightarrow \infty$ meas Letting $\delta \downarrow 0$ yields $\mu B \cap \{d(f, fn) > \varepsilon\} \rightarrow 0$; hence fn \rightarrow f. for k = 0, . The French la martingale (originally Provençal martegalo, named after the town Martiques) in equitation
means "a piece of rein used in jumping and cross country riding". By Lemma 14.30, the Chapman-Kolmogorov equation holds since (compare Exercise 14.2.1(i)) ks · kt (x, dy) = $\delta x * (N_0, s * N_0, t)$ (dy) = $\delta x * N_0, s + t$ (dy) $= \kappa s + t (x, dy)$. 1 For example, the function $f: [0, \infty) \rightarrow R, x \rightarrow 1 + x sin(x)$ is improperly Riemann 3 integrable but is not Lebesgue integrable since $[0,\infty) |f| d\lambda = \infty$. n=1 The random variables 1A (Y1), 1A (Y2), . 19.6 A tree as a subgraph of Z3 on which random walk is still transient., Xn = x) * = P $\pi Xk + 1 = x$, . If (fn) $n \in N$ converges a.e., then we also say that a.s. (fn)n \in N converges almost surely (a.s.) and write fn $\rightarrow f$. By Lemma 21.36, we have P = Q., Xk = xk] \cdot Py $\tau x1 = \infty = Px$ [X1 = x1, . Hence ν is not totally continuous with respect to μ . Clearly, fn $\rightarrow f$ pointwise; hence f is the characteristic function of a probability measure μ = w-lim μ n on R. 13.3 Prohorov's Theorem . (ii) If I is compact, then f is Hölder-continuous. Thus, for every Cauchy sequence (fN) in (Ω, d) and every $n \in N$, there exists a $qn \in \Omega$ with $N \rightarrow \infty$ dn (fN, qn) $\rightarrow 0$., n (compare (1.5)). Hence E[1A Y X]. Now we consider one model in greater detail. be probability measures $n \rightarrow \infty$ on a Polish space E with $\mu n \rightarrow \mu$. Let (Ω, A, P, τ) be the corresponding dynamical system. We show that μ (A) $\leq \infty * * n = 1 \mu$ (An). $n \to \infty n \in N$ Exercise 13.2.13 Let F, F1, F2, . Example 1.11 (i) For any nonempty set Ω , the classes A = { \emptyset, Ω } and A = 2 Ω are the trivial examples of algebras, σ -algebras and λ -systems. We can generalize this concept by allowing random times τ instead of fixed times t. For $n \in N$, we have $n1 f 1{f \ge n} \ge 1{f \ge n}$. We owe some of the proofs to [89]. 356 15 Characteristic Functions and the Central Limit Theorem 15.5 The Central Limit Theo $\tilde{\sigma} = \Lambda(t) - t \cdot x$ and thus $\Lambda^{\tilde{\sigma}} * (0) = \text{supt} \in \mathbb{R}(-\Lambda(t))^{\tilde{\sigma}}$ instead of Xi, then $\Lambda(t) = \Lambda * (x)$.) 23.1 Cramér's Theorem 591 Define $\phi(t) := e - \Lambda = \inf \phi(t)$. Definition 1.79 (Generated σ -algebra) Let Ω be a nonempty set. .) \in Cn, in \subset Bn . The characteristic 2.2 function $\phi(t) = e - \sigma t / 2$ that we get by the above calculation with t replaced by it is indeed analytic. (i) If there exists an $x0 \in E \setminus A$ such that $f(x0) = \sup f(SA(x0))$, (19.5) then f(y) = f(x0) for any $y \in SA(x0)$. Remark 16.23 If 3μ is infinitely divisible with Lévy measure ν given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then $\psi(t) := \log \operatorname{eit} x \mu(dx)$ is given by (16.22), then (16.22), then ($\pi 2 + i \operatorname{sign}(t)(c + - c -) \log(|t|)$, $\alpha = 1$ be independent random variables, that are uniformly distributed on (0, 1], i.e., Xk ~ U(0,1] for each k. If in (v) $\mu \omega = 1$ for every $\omega \in \Omega$, then μ is called counting measure on Ω . Proof (of Theorem 7.33) The idea goes back to von Neumann. (For periodic X, this is false.) Dominated convergence thus yields)) * * lim $P\pi A \in \tau - n$ (B) = E $\pi 1A \in 1\{XN = x\} \pi(\{y\})Py[B] n \rightarrow \infty x, y \in E$) * = $P\pi A \in P\pi[B]$. We will show that we can obtain a successful coupling by coalescing independent chains. 11.3 Example: Branching Process 255 Proof We compute the conditional expectation for $n \in N0$: E[Wn+1 Fn] = m-(n+1)E[Zn+1 Fn]., averaging over all permutations of X1, \bullet 18 1 Basic Measure Theory Definition 1.38 (i) A pair (Ω , A) consisting of a nonempty set Ω and a σ -algebra A \subset 2 Ω is called a measurable space. Proof The claims follow immediately from Fubini's theorem. The parallelogram law yields ; ;2 ;1 ; ; wm - wn = 2 wm - x + 2 wn - x - 4 ; (wm + wn) - x ; ; . The following theorem is of independent interest. The other cases can be proved similarly. Since every An is also open, A can be covered by finitely many An ; hence (1.13) holds. L (St1, . If $\nu \in M1(\Sigma)$, then we define the relative entropy (or Kullback-Leibler information, see [104]) of ν given μ by $\nu({x})$ H ($\nu | \mu$) := log $\nu(dx)$. Then Y is a martinga ale and $Yt \leq Xt$. Reflection For real valued random variables, a Skorohod coupling can be constructed explicitly using the distribution functions. Inductively, we define relatively compact open sets Wn \uparrow E with Wn \subset Wn+1 for all n \in N. On the other hand, if p0 = 0, then Zn is monotone in n; hence q = 0. The unique predictable process A for which (Xn2 – An)n \in I becomes a martingale is called the square variation process of X and is denoted by ()X*n)n \in I = A. 4 14.2 Finite Products and Transition Kernels Consider now the situation of finitely many σ -finite measure spaces (E, E) and (E, E) are called isomorphic if there exists a bijective map ϕ : E \rightarrow E such that ϕ is E – E measurable and the inverse map $\phi - 1$ is E - E-measurable. \bullet Example 17.19 (Computer simulation) Consider the situation where the state space E = {1, . 5.2 Weak Law of Large Numbers . + Xn } - n the number of black balls drawn in these steps. As a limit of measurable functions, If is measurable. Then $F\tau := A \in F : A \cap \{\tau \le t\} \in Ft$ for any $t \in I$ is called the σ -algebra of τ -past. Thus we can compute $\pi(VT)$ by applying (9.4) to the equivalent martingale. Consequently, the method can be efficiently implemented only if there is more structure. are independent and Poisson distributed. Show that Xn = X0 almost surely for all $n \in N0$. This can be generalized: Lemma 7.49 Let X1, X2, X3. Replace the three pairs of parallel edges with resistances 5.1 - 1.1 - 1 = 27, respectively. Theorem 7.50 Let $p \in [1, \infty)$ and assume p1 + q1 = 1. i=1 Taking the supremum over such C yields $\beta \propto i=1$ Ai $0 \mid \infty \propto Ai \leq \beta(Ai)$. ν is called totally continuous with respect to μ if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $A \in A \mu(A) < \delta$ implies $\nu(A) < \varepsilon$. For λ -systems the proof is similar. Remark 4.18 In fact, $\cdot p$ is a seminorm on Lp (μ) for all $p \in [1, \infty)$ ∞]. $n \rightarrow \infty$ (ii) For every $t \in R$, the limit $\psi(t) = \lim n(\phi n(t) - 1)$ exists and ψ is continuous $n \rightarrow \infty$ at 0. Proof Existence $x_0 \in E$ and let $E_i := y \in E : Lx_0$, y = i for i = 0, . The aim of this book is to present the central objects and concepts of probability theory: random variables, independence, laws of large numbers and central limit theorems, martingales, exchangeability and infinite divisibility, Markov chains and Markov processes, as well as their connection with discrete potential theory, coupling, ergodic theory, Brownian motion and the Itô integral (including stochastic differential equations), the Poisson point process, percolation, and the theory of large deviations. As X is transient,
we have PO [TN $< \infty$] = 1 and (as in (19.9)) * Rw, eff (0 \leftrightarrow N) Rw, eff (0 \leftrightarrow N) = . Compare Example 1.45. 4 21.3 Strong Markov Property Denote by Px the probability measure such that B = (Bt)t \geq 0 is a Brownian motion started at x \in R. Hence the logarithms could be applied on both sides. The family C = {fn , n \in N0 } separates points and is closed under multiplication; hence it is a separating class for Mf (E). Thus $\lambda(U \setminus A) < \epsilon$ for the open set U := $n \in N$ Un ., $kn+1 \leq (1 + 2\epsilon) kn+1 \leq (1$ is clear since $Cc(E) \subset Cb(E)$ and $1 \in Cb(E)$., $xN \in \{0, 1\}$ with x1 + . By Lemma 7.23, this map is also continuous. ν is called the deterministic part. $n \rightarrow \infty$ Conclude that F - 1 (u) for Lebesgue almost all $u \in (0, 1)$. For $n \in N$, define the estimated value 1 I:n := f(Xi). Definition 5.1 Let X be a real-valued random variable. B * (Rn) is called the σ -algebra of Lebesgue measurable sets. D If (i) and (ii) hold, then X λ =) λ , X* for all $\lambda \in \text{Rd}$. Theorem 20.21 Let (Xn) here X =) λ , X* for all $\lambda \in \text{Rd}$. Theorem 20.21 Let (Xn) here X =) λ , X* for all $\lambda \in \text{Rd}$. Theorem 20.21 Let (Xn) here X =) λ , X* for all $\lambda \in \text{Rd}$. $B \in \tau$ for any A, $B \in \tau$. 2 (17.9) Now assume $P[Y1 \in \{-1, 0, 1\}] = 1$ and $a \in N$. Let Π_0 , m be the set of such paths. By the definition of $F\tau$, we have $\{\tau = t\} \cap A \in Ft$ for all $t \in I$. Again probability theory comes into play when independence enters the stage; that is, when we exit the realm of linear integration theory. Summing up, we have shown the following theorem. $3 \propto \text{Let } g \in C0$ (E), $g \ge 0$. We are mostly interested in the cases I = N0, I = Z, I = [0, ∞) and I an interval. (Some authors call such a process a Markov chain in continuous time.) Let x, $y \in E$ with $x = y \in A$. Remark 2.30 The convolution is a symmetric operation: $\mu * \nu = \nu * \mu$., (N - 1)/N, 1 with transition matrix p(x, y) = bN, x $(\{Ny\})$. Then Yn \uparrow Y and Yn $E[X | F] \uparrow Y E[X | F] \uparrow Y E[X | F] \uparrow Y E[X | F] \land un m + n \psi X + Y(z) = pz + (1 - p) pz +$ Formally, both objects are of course the same. Then $\mu \partial(-\infty, x] = \mu(\{x\}) = 0$. Yn be independent expoD nentially distributed random variables with PYk = expk. Corollary 1.84 (Trace of the Borel σ -algebra) Let (Ω, τ) be a topological space and let $A \subset \Omega$ be a nonempty subset of Ω . This is the case if $E \setminus A$ decomposes into domains between which the chain that is stopped in A cannot change. On B we have $\tau n < \infty$ for every $n \in N0$ and hence *) *) P Sn = 0 infinitely often = P $\tau n < \infty$ for all $n \in N \ge P[B] = 1$. a3 a4 t 342 15 Characteristic Functions and the Central Limit Theorem summation, for all $n \in N \ge P[B] = 1$. exchangeable family of random variables X1, . & Exercise 23.2.4 Compute Λ and $\Lambda *$ in the case X1 ~ exp θ for $\theta > 0$. Then there exists a strong Markov process (Xt) t ≥ 0 . A n $\rightarrow \infty$ Exercise 13.2.12 Let X, X1, X2, . (T riangle inequality) $E[|X| F] \geq E[X|F]$. However, fy,n is a function of Yn, say fy,n = $gy, n \circ Yn$ for some map $gy, n : \{0, 1\}$ En $\rightarrow \{0, 1\}$. Chapter 12 Backwards Martingales and Exchangeability With many data acquisitions, such as telephone surveys, the order in which the data come does not matter., N, and Rn := max j = 1,..., N-1 Fn (xj -) - F \in E with P ({e}) > 0. Any such point is connected to the origin by a path without selfintersections π that starts at 0 and has length $m \ge n$. To this end, let $A \in A$ and $E \in 2\Omega$ with $\mu * (E) < \infty$. Hence the family (P{n (X) n \in N is tight. This implies that D is \-closed. Furthermore, let Sn = X1 + ..., k] > 0. Averaging over all choices i1, ... Hence we aim at finding smaller (in particular, countable) classes of sets that generate the Borel σ -algebra and that are more amenable. $\pi p({x}) = y \in E \ 466 \ 19 \ Markov$ Chains and Electrical Networks If X is irreducible and recurrent, then, by Remark 17.51, π is thus unique up to constant multiples. Indeed, evidently σ (Xn+1, Xn+2, . If x > K is a point of continuity of F, then $0 = \lim \inf Fnk(x) - \varepsilon k \rightarrow \infty = F(x) - \varepsilon \geq F(-\infty) - \varepsilon \geq F(-\infty) - \varepsilon \geq F(-\infty) - \varepsilon \geq F(-\infty) - \varepsilon \geq -\varepsilon$. (iii) A family (Xi)i \in I of random variables is called identically distributed if PXi = D PXj for all i, $j \in I$. 558 21 Brownian Motion Remark 21.49 (i) By using higher moments, it can be shown that the paths of Y are Höldercontinuous of any order $y \in (0, \infty)$ 12). , gm , g1) is a self-avoiding path starting and ending in g1 \in TL. Furthermore, we construct measures, in particular probability measures, on such classes of sets. Here we cite without proof a theorem that was found independently by Rademacher [141] and Menshov [113]. Define the time of first excess of a (truncated at (n + 1)), $\tau := \inf\{m \ge 0\}$: $Xm \ge a$ \land (n + 1). The momentousness of the following concept will become manifest only gradually. This dual space is defined as follows. It is based on a diagonal sequence of zeros and ones that when concatenated vield the message. (i) Assume that X is a real random variable and that $(Xn) \in N$ is a subsequence such that $l \to \infty PXn - PX$ weakly. $ik \leq n$. E[F (X1, ..., (i) Assume there exists a T $\in N$ with $\tau \leq T$. i=1 n We call u =: ui the product measures of the measures $u_1 \dots u_i$]. Bn $\in B(E)$ as well as A = ni=0 Xt-1 (Bi). (ii) If dx = dy for all u =: ui the product measure of the measures $u_1 \dots u_i$]. $x, y \in E$, then d := dx is called the period of X. Let $A \in A$ with $\mu(A) = 0$. We distinguish two cases: Case 1: t < n-1. Then $\sigma E = \sigma(E)$. Then X is also a (sub-, super-) martingale with respect to the smaller filtration F. p q Proof Fix $y \in [0, \infty)$ and define f (x) := Theorem 7.16 (Hölder's inequality) Let $p, q \in [1, \infty]$ with $Lp(\mu), q \in Lq(\mu)$. Varadhan's Lemma and Free Energy 603 The method of the proof that we applied in the last example to derive the LDP with rate function I⁻ is called a contraction principle. In this case, p is not aperiodic. Hence, the Borel σ -algebra equals the generated λ -system: B(Rn) = $\delta(E_1)$ for i = 1, 2, 3, 5, . Rw + Rw Rw + Rw - = R + = ∞ , then X is recurrent and hence every point is visited (ii) If Rw w infinitely often. By construction, μ (U) < ∞ ; hence 1U \in L (μ i) for i = 1, 2, m and tl - tl-1 \leq 2-l for l = n, 1! (n - 1)! n! Proof As the nth derivative of eit has modulus 1, this follows by Taylor's formula. Similarly as in Remark 17.28, we define independent random variables T1, T2, The main goal is the representation theorem for continuous linear functionals on Hilbert spaces due to Riesz and Fréchet. Then the sequence X1, X2, For the convenience of the reader, we recall the definition of a topology. (5.19) 5.5 The Poisson Process 143 This is equivalent to showing that for each choice of k1, On the other hand, if there was a self-avoiding path $(q0, . Then \kappa 1 \cdot \kappa 2 = \mu * \nu$. Therefore, $P[\xi_n(X)(K c) > \epsilon] \le \epsilon - 1 E[\xi_n(X)(K c)] = \epsilon - 1 P[X1 \in K c] \le \epsilon$. Proof Let $En \uparrow \Omega$ with $\nu(En) < \infty$, $n \in N$. Let (X, Y) be the coupling that was constructed in Step 2 and let $\tau := \inf n \in N0$: Xm = Ym for all $m \ge n$. be independent, exponentially distributed random variables with PTn = expn2. Therefore, there is a constant C = C(N, x) such that) n * Ex (Z⁻ s+t - Z⁻ sn)4 $\leq C t 2$ for all s, $t \in [0, N]$ with t E[X1], lim $n \rightarrow \infty$) * 1 log P Sn $\geq xn = -I(x) := -\Lambda * (x)$. ---. To this end, we first introduce the notion of the tail σ -algebra. (2.8) The case J = J is exactly the claim we have to show. If p is irreducible and aperiodic, then $|\lambda 2| < 1$. Furthermore, $\infty = \epsilon 2 - n = \epsilon$. The 3n paths leading from the nodes of the (n+1)th generation to those of the (n+1)th generation are disjoint paths, each of length 2n-1. The sequence is called subadditive, if $am + n \le am + an$ for all $m, n \in \mathbb{N}$. Each of the offspring gets its own independent exponential lifetime. If we $x \to \infty$ only have $F(\infty) \le 1$ instead of $F(\infty) = 1$, then F is called a (possibly) defective p.d.f. If μ is a (sub-) probability measure on R, B(R), then F μ : $x \rightarrow \mu((-\infty, x])$ is called the distribution function of μ . It is enough to show that there exists a sequence (H n) n \in N of bounded, progressively measurable processes such that (25.3) holds. Which condition of the theorem would be violated? θ -t Hence the distribution of X. is characterized by its moments. + Dn for $n \in N0$. Then the value of our portfolio is described by a discrete stochastic integral which is again a martingale. This fact is the basis for nonparametric tests on the equality of distributions. As $x \to d(x, Aci)$ is continuous for i = 1, 2, the closed subsets B1 and B2 of C are compact. If we define $S^{-1} := \inf\{Sn : n \in N\}$ $\in N$, then indeed *) * P S^{*} = 1 - 2n - 1 = P D1 = . $\in E$ in such a way that [ω 1, . Proof For any $n \in Z$, PX+Y ({n}) = P[X + Y = n] + , {X = m} \cap {Y = n - m} = P m
\in Z =) * P {X = m} \cap {Y = n - m} = P m (X = m) \cap {Y = n - m} = P m 19.2 Reversible Markov Chains. In addition, we have $An = B \in Pn B \subset An Pn - := \{B \in Pn : \phi(B) < 0\}$, $Pn + := Pn \setminus Pn - and Cn := B$. This is the coarsest topology such that for all $f \in Cb$ (E), 3 the map Mf (E) $\rightarrow R$, $\mu \rightarrow f$ du is continuous. We knew this already from Theorem 9.35; however, here we could also quantify how much f (X) differs from a martingale. (iv) lim inf μ n (E) $\geq \mu$ (E) and lim sup μ n (F) $\leq \mu$ (F) for all closed F \subset E. Takeaways Weak convergence of measures is defined via convergence of measure and • ϕ is even (that is, $\phi(x) = \phi(-x)$). In the terminology of statistical physics this means that the time average, or path (Greek: odos) average, or We define θ (p) := P[#C p (0) = ∞] as the probability that the origin is in an infinite open cluster.) λ , X* ~ N) λ , μ *, λ , $C\lambda$ * for every $\lambda \in \text{Rd}$..}) < 4 ϵ . properties: X is a modification of X & Exercise 21.4.4 Let X be a stochastic process with values in a Polish space E and + with RCLL paths. At each step, the walker chooses one of the adjacent open edges at random (with equal probability) and traverses it. Hence we restrict ourselves to the case n = 2. 3 f dµ and 3 g dµ is Proof (i) 3Clearly, $f + \leq g + a.e.$, hence $(f + -g +) + d\mu = 0$. We write • An \uparrow A and say that (An)n \in N increases to A if A1 \subset A2 \subset . + xn . It is easy to check that the map κ : Rd × B(Rd $|\otimes[0,\infty)|$, $(x, A) \rightarrow Px[A]$ is a stochastic kernel. The idea is simple. (ii) What is the probability that the kth person gets his or her reserved seat? Lemma 23.12 For every $n \in N$ and $\nu \in En$, we have $(n + 1) - \#\Sigma e - n H(\nu \mid \mu) \leq P[\xi n(X) = \nu] \leq e - n H(\nu \mid \mu)$. The space (Ω, d) is separable and hence Polish. By the monotonicity principle, the effective resistance from 0 to ∞ can be bounded by Reff (0 $\leftrightarrow \infty$) = lim Reff (0 $\leftrightarrow \{-n, n\}$) $n \rightarrow \infty = \infty$ 1-p n n=0 p = p < ∞ . Let X and Y be Rn -valued random variables with densities fX and fY. (iii) μ is called \emptyset -continuous if (ii) holds for A = \emptyset . 163 163 165 172 175 179 186 8 Conditional Expectations. In particular, for y > 0, n \in N and $k \in \{1, . Definition 20.24 \text{ A measure-preserving dynamical system } (\Omega, A, P, \tau) \text{ is called mixing if } * \text{ lim P A} \cap \tau - n (B) = P[A] P[B] n \rightarrow \infty \text{ for all } A, B \in A. However, the distributions of Y and Z do not coincide. Then } (X Y) \in L1 (P) \text{ and } E[XY] = E[X] E[Y].$ \in F. of random variables. Since F 7.6 Supplement: Dual Spaces 189 Remark 7.51 For p = ∞ , the statement of Theorem 7.50 is false in general. Then X : $\Omega \rightarrow \Omega$ is A - 2 Ω -measurable if and only if X-1 ({ ω }) \in A for all $\omega \in \Omega$..., we have $\omega i \rightarrow 1A$ ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) $\mu 3-i$ (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ 1An ($\omega 1$, $\omega 2$) (d $\omega 3-i$) = ∞ magnetization m = n1 ni=1 of describes the state of the system completely (as the particles are indistinguishable). Why? Clearly, An is independent of Fin and thus *) PAn Fin = P[An] > \epsilon. If (Xn)n \in N0 is a stochastic process with distributions (Px , x \in E), then the Markov property in Definition 17.3(iii) is implied by the existence of a stochastic kernel $\kappa 1 : E \times B(E) \rightarrow [0, 1]$ with the property that for every $A \in B(E)$, every $x \in E$ and every $s \in I$, we have *) Px Xs+1 $\in A$ Fs = $\kappa 1$ (Xs, A). Note that at this point it is not even clear that the paths are measurable maps. ($\omega 1, ..., \omega i$) $\in E i - 1 \times A^{\sim} i$ Intuitively, the family (Ai)i $\in N$ should be independent if the definition of independence makes any sense at all. $\star x \rightarrow \infty$ We will now have a closer look at the case where μF is a probability measure. We say that $\sigma(X) := X - 1$ (A) is the σ -algebra on Ω that is generated by X. 15.3 Lévy's Continuity Theorem 23 where inf $\emptyset = \infty$. Then the following are equivalent: (i) X is a Brownian motion. For any bounded measurable function ϕ : E n \rightarrow R and for any \in S(n), we have $E[\phi(X) A] = E[\phi(X) A] = E[\phi(X) A] = E[\phi(X) A] = E[\phi(X) A]$ X)] exists, then ϕ is k-times differentiable at t with $\phi(k)(t) = \phi k(t)$. n=1 (5.7) 126 5 Moments and Laws of Large Numbers Thus, by the Borel-Cantelli lemma, for P-a.a. ω , there is an n0 = n0 (ω) such that Skn - n/4
- E[X] 1 < (1 + ε) k n for all $n \ge n0$, whence lim sup kn-1 Skn - E[X1] = 0 almost surely. Hence Monte Carlo simulation should be applied only if all other methods fail. k=1 Birkhoff's ergodic theorem yields lim inf $n \rightarrow \infty 1$ Rn $\geq P[A|I]$ a.s. n (20.5) For the converse inequality, consider Am = {Sl = 0 for l = 1, . k n} Let $\mu k \in M1$ (R) be the distribution on R with density $\pi 1 - \cos(a)$. Then the following three statements are equivalent. Woyczy nski Case Western Reserve University Universitext is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. However, clearly there is no sequence (bn $n \in N$) = am bm., n, n $\in N$) by Xn, l = $n \rightarrow \infty$ (Yl – E[Yl])/ σn . 23.3). In particular, we define Nt := N(0,t] as the total number of clicks until time t.

Clearly, $\mu(\{0\}) = 0.22$ Note that this is the rate function from Theorem 23.1. \blacklozenge Next we describe formally the connection between the LDPs of Sanov and Cramér that was indicated in the previous example. The infinite series is a Gaussian process with the same covariance function as Brownian motion. (ii) For $x \in I \circ$, define the function of difference function of difference function as Brownian motion. quotients gx (y) := $\phi(y) - \phi(x) y - x$ for $y \in I \setminus \{x\}$. By the Portemanteau theorem (Theorem 13.16(iv)), for any $N \in N$, $\mu(Acn,N) \ge \lim \inf \mu Nk$ (Acn,N) ≥ 0 , $\omega \to d(f(\omega), g(\omega))$ is A - B([0, ∞)). measurable. Then $\nu(A) = F(1A)$ is a signed content on A and we have $|\nu(A)| \leq F p(\mu(A))1/p$. For the latter property, it is not sufficient that the chain be irreducible. Upper semicontinuity of μ implies $n \rightarrow \infty$ μ Dn (ϵ) $\cap A = 0$ for any $A \in A$ with $\mu(A) < \infty$. n - 1 k = 0 In particular, if τ is ergodic, then 1 $n - 1 n \rightarrow \infty Xk \rightarrow E[X0]$ in Lp (P). (20.1) Remark 20.2 If I = N0, I = N or I = Z, then (20.1) is equivalent to L [$(Xn+1)n\in I$] = L [$(Xn)n\in I$]. Note that $j\in J\times E = jj\in J(X_1)-1$ (Ej) $\in AJ$, $j\in J$ hence AJ $\subset AJ$. 391 399 404 411 415 423 429 18 Convergence of Markov Chains. k=1 Theorem 14.32 (Fubini for transition kernels) Let (Ωi , Ai) be measurable spaces, i = 1, 2. Am $\leq \lim m = n$ $n \rightarrow \infty \infty$ P[Am] = 0. Definition 8.24 Let $Y \in L1$ (P) and $X : (\Omega, A) \rightarrow (E, E)$. In the first section, we give the basic definitions and derive simple properties. Then $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = C hence, by assumption, $\{|f| > h\}$ $\delta(\varepsilon)$ h dµ = Nt for t = T and let N[~] T = 0. Assume that 21.9 Pathwise Convergence of Branching Processes 559 (Zin)i \in N is a sequence of Galton-Watson processes 559 (Zin)i \in N, and let Hn = 1{n=n0} . If $p \in (0, 1)$, then Berq 0 Berp . \blacklozenge Example 12.16 (Strong law of large numbers) If Z1 , Z2 , . If $\beta > 1$, then (23.23) β , 0 β , 0 β , 0 β , 0 has two other solutions, $m - \in (-1, 0)$ and m + = -m -, whose values can only be computed numerically (Fig. 14.2 Finite Products and Transition Kernels . Proof This is an immediate consequence of Theorem 24.5. Definition 24.10 Let $\mu \in M(E)$. We start with a lemma that is due to Varadhan [167]. Definition 6.8 (Mean convergence) Let f, f1, f2, . Theorem 21.17 (Paley-Wiener-Zygmund [126]) For every $\gamma > 12$, almost surely the paths of Brownian motion (Bt)t ≥ 0 are not Hölder-continuous of order γ at any point. Show that p is the transition matrix of an irreducible, aperiodic random walk and compute the invariant distribution and the exponential rate of convergence. Proof Let N $\subset \Omega$ be a null set such that fn (ω) \uparrow f (ω) for all $\omega \in N$ c. Applying Prohorov's theorem (i.e., Corollary 13.30) to the measures ($\mu k \ 1W \ n$) $k \in N$ and sets $Ck, 1, .., Xn = x \ |Xk = x \ P\pi \ [Xk = x]$) $* = \pi(\{x\}) \ Px \ X1, ..$ Lemma 1.52 If $\mu *$ is an outer measure, then $M(\mu *)$ is a σ -algebra. Remark 13.27 If E is Polish, then by Lemma 13.5, every singleton $\{\mu\} \subset Mf(E)$ is tight and thus so is every finite family. (i) Show the chain rule for the Radon-Nikodym derivative: $d\nu d\mu d\alpha$ (ii) Show that $f := d\nu d(\mu + \nu)$ exists and that $d\nu d\mu \alpha$ -a.e. = f 1 - f holds μ -a.e. $\Rightarrow 7.6$ Supplement: Dual Spaces By the Riesz-Fréchet theorem (Theorem 7.26), every continuous linear functional $F : L2 (\mu) \rightarrow R$ has a representation F (g) = f, g^* for some $f \in L2 (\mu)$. Now fix $\omega \in \Omega \setminus N$ and choose $n0 = n0 (\omega)$ such that ∞ An . , Xn-k = x) * = $\pi(\{x\})$ Px $\tau x1 \ge n - k + 1$. For nonnegative functions, the Lebesgue integral can be computed via a kind of partial integration formula (Theorem 4.26). Assume that for i = 1, . At which point would the proof fail? Show the statement analogous to Exercise 8.3.6. Show that (R, B(R)) and Rn, B(Rn) are isomorphic. Hence *) *) *) E Xo 1{o m - 1}, k=1 and each of the events is in Fm-1. Define $qn := f(b) 1{b} + n n n, tn)$, (inf f ([ti-1, tin)) 1[ti-1ii=1 hn := f (b) $1\{b\} + n \text{ nn}, tn$). $n \rightarrow \infty$ (21.33) and Theorem 21.38, it is enough to show that ($L[S^n], n \in N$) is tight. As ψ is strictly convex and since $\psi(0) \ge 0$, there is a unique $r \in [0, 1)$ such that $\psi(r) = r$., Dn-1 only. Show for any $\varepsilon > 0$, there is an $A \in A$ with $\mu(A) < \infty$ and $A f d\mu - f d\mu < \varepsilon$. Rw + Rw $- = \infty$ and $R + = \infty$, then $\lim \inf X = -\infty$ and $\lim \sup X = \infty$ (ii) If Rw n n w n $\rightarrow \infty$ almost surely. The a.s. $\lim W \infty = \lim W$ n exists and $n \rightarrow \infty$ m $>1 \iff E[W \infty] > 0$. $n=1 * Let A \in A$. F is called symmetric if F is n-symmetric for all $n \in N$. Hence, for a > 0, $\sup f \in F \{|f| \ge a\}$ H (|f|) d $\mu = 1$ sup Ka $f \in F \{|f| \ge a\}$ H (|f|) d $\mu = -\infty$ H $(|f|) d\mu \rightarrow 0.$ (iii) $(\sigma \cup \text{-stability})$ Let A1, A2, be probability distribution functions on R, and $n \rightarrow \infty$ assume Fn \Rightarrow F. We conclude that $h(P, \tau r) = h(P, \tau r; P) = 0$. Then X -1∞ n=1 hence ∞ n=1 An $= \infty$ X-1 (An $) \in \sigma$ (X-1 (E)); n=1 An $\in A0$. $\in 2E$. in turn, however, are convex. (ii) We define the indicator function on the set A by 1A (x) := 1, if $x \in A$. In fact, we will show a slightly stronger statement in Theorem 1.53. It remains to show that ft is continuous at 0. Recall that $N = \{f \in L2(\mu) : f(x) \in A, 0, if x \in A, if x$ $(-\infty) = 0$. Denote by ' (LX (f) = E exp - f dX, f \in B + (E), the Laplace transform of X and by ' (ϕX (f) = E exp i f dX, f \in BbR (E), the characteristic function of X. In this case, the formulas for the second moments of sums are particularly simple. Thus fz + gz and H = infz \in F (fz + gz) are measurable. This the time of the nth click. Later we will see that the assumption that E is finite can be dropped. + Yn2. Now, for any i \in I, let β i = {B ϵ (xi) : xi \in Di , $\epsilon \in Q+$ } be a countable base of the topology of Ω i consisting of ϵ -balls. Clearly, X and X⁻ have the same harmonic functions. n $\rightarrow \infty$ D (ii) Fn $\rightarrow \neq F$. Hence Theorem 2.16 yields the claim. Here R1 = 1, R2 = 2, R3 = 1, δ = 5, R 2 = $\delta/R2 = 5/2$ and R 3 $= \delta/R3 = 5.7.5$ Supplement: Signed Measures. Indeed, if 1 1 ϵ An := n Sn > ϵ and A = lim sup n Sn > 0, then clearly $n \rightarrow \infty A = m \in N 1/m$ lim sup An $n \rightarrow \infty$; (' hence P lim sup An $n \rightarrow \infty$; (' hence P lim sup An $n \rightarrow \infty$; (' hence P lim sup An $n \rightarrow \infty$; (' hence P lim sup An $n \rightarrow \infty$; (' hence P lim sup An $n \rightarrow
\infty$; (' hence P lim sup An $n \rightarrow \infty$; (' hence P lim sup An a.s., then there is a random variable Y with values in $\{-1, +1\}$ and with E[Y|X] = X. This shows (LDP 2). Successively draw without replacement all of the balls and define Xn := 1, if the nth ball is black, 0, else. Together with z = x, it follows that d (Lx, y + Ly, x). 20.3 Examples Example 20.17 Let (X, (Px)x \in E) be a positive recurrent, irreducible Markov chain on the countable space E. Then $\mu(N) = 0$ and $n \rightarrow \infty$ fn (ω) $- \rightarrow$ f (ω) for any $\omega \in \Omega \setminus N$. Let |X| = M + A be Doob's decomposition of |X|. Hence there exist En \uparrow E with $\mu(En) < \infty$ for every $n \in N$., hn) starting and ending in some point h0 = hn = x \in TL. Also L1 -convergence implies stochastic convergence. \clubsuit 1.2 Set Functions We aim at assigning to each "event" (which will be formalised later) a number that can be interpreted as the probability for the event to occur. Furthermore, given Z, the sequence X1, X2, . We can weaken the condition in Theorem 5.16 in a different direction by requiring integrability only instead of square integrability of the random variables. Hence, A \in Fo $C F\tau + .$ Proof It suffices to check that the Chapman-Kolmogorov equation $\kappa t \cdot \kappa s = \kappa s + t$ holds. (ii) By (i), we have $\lim n \to \infty$ fn $d\mu = \sup fn d\mu \leq f d\mu$. (i) The family (P[X0i $\in \cdot$], $i \in I$) of initial distributions is tight. To this end, consider independent random variables T1s, T1r, T2s, T2r, Example 2.17 Let E be a countable set and let (Xi) i \in I be a count random variables with values in (E, 2E). Now consider K = C. \neq r d λ d (C) r d λ d (C) r d λ d (C) Exercise 13.1.8 Similarly as in Corollary 13.7, show the following: Let E be a σ compact polish space and let μ be a measure on E. Exercise 2.4.1 Let T be the infinite binary tree (Fig. Theorem 5.29 Let X1 , X2 , . To this end, let (ν t) t \geq 0 be a convolution semigroup on Rd and let κt (x, dy) = $\delta x * \nu t$ (dy). Show that $\mu \epsilon := 12 \text{ N} - 1, \epsilon + 12 \text{ N} 1, \epsilon$ satisfies an LDP with good rate function I (x) = 12 min((x + 1)2). (i) If we assume that for any i = 1, 2, 3 the event Ai dependent. 4.2 Monotone Convergence and Fatou's Lemma What are the conditions that allow the interchange of limit and integral? As f is harmonic on E \ A, we have pA f = f on E. Until now, we have not assumed that X is a martingale. Define ν^{n} n \in M1 ([0, ∞)) –x by ν^{n} n (dx) := 1 – e un (1) ν n (dx). 2, here the random variables need not be independent., n} \ J. Assume that one of the following conditions holds. 9–we have not assumed that X is a martingale. 20, and 23. Let $\phi k (x_1, h_1 (\omega_1)h_2 (\omega_2)$ Hence, it is enough to show the statement for the finite measures $\mu^2 i := h_1 \mu_1 instead$ of $\mu_1, i = 1, 2$. As an application of the individual random of the individual random of the statement for the finite measures $\mu^2 i := h_1 \mu_1 instead$ of $\mu_1, i = 1, 2$. As an application of the individual random of the statement for the finite measures $\mu^2 i := h_1 \mu_1 instead$ of $\mu_1, i = 1, 2$. As an application of the individual random of the statement for the finite measures $\mu^2 i := h_1 \mu_1 instead$ of $\mu_1, i = 1, 2$. As an application of the individual random of the statement for the finite measures $\mu^2 i := h_1 \mu_1 instead$ of $\mu_1, i = 1, 2$. As an application of the statement for the finite measures $\mu^2 i := h_1 \mu_1 instead$ of $\mu_1 instead$ of $\mu_1 i := h_1 \mu_1 instead$ of $\mu_1 i := h_1 \mu_1 instead$ of $\mu_1 i := h_1 \mu_1 instead$ of $\mu_1 instead$ of $\mu_$ walk step is $\sigma 2 := 4r(1 - r)$. Clearly, (Y1, ..) $\in A$. (21.18) 530 21 Brownian Motion Furthermore, $F\tau n \downarrow F\tau + := F\sigma \supset F\tau$. Remark 17.4 We will see that the existence of the transition kernels (kt) implies the existence of the transition kernels (kt) implies the existence of the kernel k. Now we want to compute the price of a European call option VT := (XT - K) + explicitly. It is conjectured that θ (pc) = 0 holds in any dimension $d \ge 2$. k The other inclusions Ei $\subset \sigma$ (Ej) can be shown similarly. Let $A \subset E$ be such that $A = \emptyset$ and $E \setminus A$ is finite. $n \rightarrow \infty$ Hence PNt = Poi α t. For $f \in L1$ (μ), we define the integral of f with respect to μ by $f(\omega) = f d\mu = f d\mu - f - d\mu$. (v) ϕ is differentiable at x if and only if $D - \phi(x) = D + \phi(x)$. In that form, the theorem goes back to Choquet and Deny [24], see also [144]. The situation becomes even more puzzling if we restrict the random walk to, e.g., the upper half plane {(x, y) : x ∈ Z, y ∈ N0 } of Z2 . If there exists a successful coupling, then every bounded harmonic function is constant. motivation, consider the following example. The Fig. If K = C, then in addition assume that C is closed under complex conjugate function f is also in C). We can assume that $\psi = 1B$ for some $B \in T$., n - 1. Thus the random walk with weights C is reversible. (iv) Consider the infinite product measures (see Theorem 1.64) (Berp) N and (Berg) N and (restriction of μ to Z R. 20.2 Ergodic Theorems . k, With this definition, 1 Cw (i, i + 1) = = wi + Cw (i) i + 1 i Cw
(i, i - 1) = = wi - As most applications only need (i) \Rightarrow (ii), we only prove that implication. From this (12.2) follows. Example 9.41 Consider the very simple martingale X = (Xn)n=0,1 with only two time points. & Exercise 4.3.3 If f: [0, 1] \rightarrow R is Riemann integrable, then f is Lebesgue measurable. We therefore obtain the rather intuitive statement that as $n \rightarrow \infty$ the distributions of k-samples with replacement, respectively, become the same: ; ; lim sup ; μ , k (x) – ν n, k (x); T V = 0. y \in Zd {#C p (y) = ∞ }. 29 Clearly, the latter computation is more complicated than using the resistances R from the reduced network directly. 7.5 Supplement: Signed Measures 183 n = C \cup . The distribution of a random measure is characteristic function as well as by its characteristic function. The network directly is characteristic function as well as by its characteristic function as well as by its characteristic function as well as by its characteristic function. 1. 4 212 8 Conditional Expectations Exercise 8.3.7 Let E be a Polish space and let P, Q \in M1 (R). a 4.3 Lebesgue Integral Versus Riemann Integral Versus Riemann Integral 109 Example 4.24 Let f: [0, 1] \rightarrow R, x \rightarrow 1Q. Hint: Use a similar argument as in the proof of Theorem 2.45. (2.14) The fundamental question is: How large are θ (p) and ψ (p) depending on p? < tn , we have that $(X_{ti} - X_{ti} - 1)_{i=1,...,n}$ is independent, (iii) a Gaussian process if X is real-valued and for all $n \in N$ and t_1 , Now the first inequality of (4.7) follows from f $d\mu = \infty \mu(\{f \ge n\}) = n = 1 \infty \mu(\{f \ge n\})$. Furthermore, $x = X_0 \in \{0, 1\} \Lambda$ is the initial state. Hence (P2) holds. $= 1 - \cos(x) \pi 2$ Exercise 16.2.3 Let Φ be the distribution function of the standard normal distribution N0.1 and let $F: R \rightarrow [0, 1]$ be defined by $21 - \Phi x - 1/2$, if x > 0, F(x) = 0, else. Particularly helpful is a moment criterion that postulates that moments of increments over small intervals decay guickly as the intervals get smaller. Thus $A \in [0, 1]$ be defined by $21 - \Phi x - 1/2$, if x > 0, F(x) = 0, else. $F\tau$. Then we present the theorem that justifies our hope. To this end, we extend the corresponding theorem (C) ≥ 1 im inf μ (C) ≥ 1 im inf μ (C) ≥ 1 im inf μ (C) $\geq \mu$ (C) $-\epsilon$. Thus, by the Proof By Theorem 4.26, we have n=1 Borel-Cantelli lemma,) * P Xn = Yn for infinitely many n = 0. (21.10) In other words, for dyadic rationals D, X(ω) is (globally) Hölder- γ -continuous. Let m := E[X1,1] < ∞ be the expected number of offspring of an individual and let σ 2 := Var[X1,1] \in (0, ∞) be its variance. By the monotone convergence theorem, we get n $\rightarrow \infty$) * α t = E [Nt] = lim E Ntn = lim pn 2n . (iv) Let $\theta > 0$ and let X be exponentially distributed, X ~ exp θ . 2.4 Example: Percolation 73 Exercise 2.3.1 Let (Xn) n \in N be an independent family of Rad1/2 random variables (i.e., P[Xn = -1] = P[Xn = +1] = 12) and let Sn = X1 + . Then there are $\varepsilon > 0$, $f \in Cb$ (E) and (nk) k \in N with nk $\uparrow \infty$ and such that $f d\mu n - f d\mu > \varepsilon k$ for all $k \in N$. In prose, almost surely eventually only balls of one color will be drawn. Thus (Xi, j, (i, j) \in I × I) is uniformly integrable by Theorem 6.19. More precisely, up to the first summand, it is the Karhunen-Loève expansion of the Brownian bridge (Xt - tX1) t \in [0,1] (see, e.g., [1, Chapter 3.3]). k k=0 (x) (Negative binomial distribution) By the generalized binomial theorem (Lemma 3.5), for all $x \in C$ with |x| < 1, $(1 - x) - r \propto -r = (-x)k$. Denote by $c(e) \in \{0, 1\}$ (e) the code of e, where l(e) is its length. (i) Show that $P[\{\Phi \in \cdot\} | \Theta = \theta]$ for almost all θ has the density 14 | $cos(\phi)$ | for $\phi \in [-\pi, \pi]$. Using (19.13), the probability that the random walk visits 1 before 0 is P = 27 32 27 32 + 27 26 = 13. For $n \in N$, denote by Pnthe projection of P on E n = E {0,...,n-1}; that is, ('Pn ({(e0, . Using Fatou's lemma, we infer E[Xt] $\leq \lim \inf E[Xt \land n] \leq E[Xt] < \infty$. Further, let $\varphi : E \to R$ be continuous and assume that inf lim sup $\varepsilon \log M > 0 \varepsilon \to 0$ eq(x)/ $\varepsilon 1$ { $\varphi(x) \geq M$ } $\mu \varepsilon (dx) = -\infty$. While these theorems work with real random variables, we will also see limit theorems where the random variables take values in more general spaces such as the space of continuous functions when we model the path of the random motion of a particle. Klenke, Probability Theory, Universitext, 327 328 15 Characteristic Functions and the Central Limit Theorem exp(z1) · exp(z2). Example 1.40 (Product measure, Bernoulli measure) We construct a measure for an infinitely often repeated random experiment with finitely many possible outcomes. * Exercise 5.3.3 Let E be a finite set and let p be a probability vector on E. 1.2 Set Functions 17 Hence μ is σ -subadditive. Compute the entropy of the bivariate chain on E1 × E2 with transition matrix p given by p((x1, x2), (y1, y2)) = p1 $(x_1, y_1)_{p2}(x_2, y_2)$. $n \rightarrow \infty$ (ii) Fn \Rightarrow F if and only if $d(Fn, F) \rightarrow 0$. Define a content μ on $A = \{[\omega_1, . \bullet Example 20.10 \text{ Let } X = (Xn)_{n \in \mathbb{N}} | \psi(A \cup B) = \mu * (A \cap (A \cup B)) = \mu * (A$ In the case In the case $\alpha \ \alpha \ \alpha \in (0, 1)$, let $bn \equiv 0.662$ Independence Proof For $k \in K$, let Zk = 0.1 Aj : Aj $\in \sigma$ (Xj), #{j $\in Ik$: Aj $= \Omega$ } < $\infty j \in Ik$ be the semiring of finite-dimensional rectangular cylinder sets. Reflection Check that nearest neighbour random walk on the random subgraph of open edges. The effective resistance from 0 to ∞ can be computed by the formulas for parallel and sequence connections, ∞ Reff (0 $\leftrightarrow \infty$) = 1 R(i, i + 1) = ∞ . 17.1 Definitions and Construction 397 Using the tower property of the conditional expectation and Theorem 17.9 in the third equality, we thus get) *) * Ex f (XT + t) t \in I $TT = 1{T = 1}{T = 1{T = 1}{T = 1{T = 1}{T = 1}{T = 1}{T = 1}{T = 1}{T = 1{T = 1}{T = 1}{T$ =s} Ex f (Xs+t)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXs f (Xt)t \in I τ s \in I = ') * (Ex 1{ τ =s} EXt f (Xt)t \in I τ s \in I τ = ') * (Ex 1{ τ =s} EXt) = (Ex 1{{}\tau =s} EXt) = (Ex 1{ τ =s} EXt) = (Ex 1{{}\tau =s} EXt) = (Ex 1{}\tau =s} E assume that X takes values in E := [0, 1]. Up to the order, the resulting distribution is thus the generalized hypergeometric distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; that is, $P[Nt - Ns = k] = e - \alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; the difference Nt - Ns is Poisson-distributed with parameter $\alpha(t - s)$; the difference Nt - Ns is Poisson-distributed Nt + Ns. CPoivn is infinitely divisible, by Corollary 16.9, the weak limit is also infinitely divisible. Define f (x) := $g(x_{\tau})$, if $x \in A$, (19.1) if $x \in E \setminus A$. Assume that E[X1] = 0 and P[X1 = 0] < 1. Rephrased to the language of financial markets this means: If the price of a risky asset is given by a binary splitting process than there is a hedging strategy for any contingent claim. Let X0, X1, Definition 14.49 (Convolution semigroup) Let I $\subset [0, \infty)$ be a semigroup. + xk for k = 0, Hence; n; X - Xn-1; $\leq 2-(n+1)/2$ max [$\xi_{n,k}$ |, k = 1, Proof We follow the exposition in Dieudonné [34,
Chapter VII.3]. Then we have equality in (17.8). If you're looking on a site with a map function, you may also see a map with the location pinned and an option to get turn-by-turn directions to the place you're calling. Reverse Phone Number Lookup A reverse phone number and want to know who it belongs to before you call. In this case, uniqueness of the market ("the market ("the market to completeness of the market to completeness of the market to completeness of the market ("the market to completeness of the market ("the market fundamental theorem of asset pricing" by Harrison-Pliska [68]). We consider (wi-)i $\in Z$ as an environment in which X walks and later choose the environment in which X walks are choose Probability Measures on $C([0, \infty))$ Let X and $(Xn)n \in N$ be random variables with values in $C([0, \infty))$ (i.e., continuous stochastic processes) with distributions PX and $(PXn)n \in N$ and x > K. (8.14) 206 8 Conditional Expectations By construction, $F^{\sim}(\cdot, \omega)$ is monotone for all $n \in N$ and x > K. (8.14) 206 8 Conditional Expectations By construction, $F^{\sim}(\cdot, \omega)$ is monotone for all $n \in N$ and x > K. increasing and right continuous. Then there is an $A \in A$ with $\mu(A) > 0$ and an $\varepsilon > 0$ with $\varepsilon (x) = nU(x)$, where U(x) is the average energy of one particle of the ensemble in state x. Then, by Theorem 5.6(i), $0 \leq Var[X + \theta Y] Var[Y] = nU(x)$, where U(x) is the average energy of one particle of the ensemble in state x. Then, by Theorem 5.6(i), $0 \leq Var[X + \theta Y] Var[Y] = nU(x)$, where U(x) is the average energy of one particle of the ensemble in state x. Then, by Theorem 5.6(i), $0 \leq Var[X + \theta Y] Var[Y] = nU(x)$. $Var[X] + 2\theta Cov[X, Y] + \theta 2 Var[Y] Var[Y] = Var[X] Var[Y] = Var[Y] Var[Y] Var[Y] = Var[Y] Var[Y] Var[Y] = Var[Y] Var[Y] Var[Y] Var[Y] Var[Y] = Var[Y] Var[Y]$ (but not the obligation) to buy one stock at time T at price K (from the issuer of the option). 21.8 Donsker's Theorem 19.2 (Superposition principle) Assume f and g are harmonic on E \ A and let α , $\beta \in \mathbb{R}$. Further, let $T = \infty$ n=1 σ (Xm , m \geq n) be the tail σ -algebra of X1 X^2 , Further, let g: I \rightarrow R, i \rightarrow E[X |Bi]. In the following, let X, X1, X2, (27 + 1)-1 = 27 (25 = 27 and (19 + 125 32, 54 + 2) 26 513) 8 In the reduced network, we have the resistances R (0, x) = 27 32 and R (x, 1) = 27 26. Hence the "conditional distribution of (Y1, . n $\rightarrow \infty$ n $\rightarrow \infty$ 4.2 Monotone Convergence and Fatou's Lemma 105 Proof By considering $(fn - f)n \in N$, we may assume $fn \ge 0$ a.e. for all $n \in N$. - = CPoi for some $\nu \in M$ (N). On the edge set E, define the translation $\tau : E \rightarrow E$ by τ ()x, y*) =)x + u1 , y + u1 *. Theorem 5.6 Let $X \in L2$ (P). Here F ((x1 , x2)) := min(F1 (x1), F2 (x2)) defines a distribution function on $R \times R$ (see Exercise 1.5.5) that corresponds to a coupling ϕ with $\phi(L) = 1$. Water can flow only through the remaining tubes. If there exists a sequence TN $\rightarrow \infty$ with E[XTN] \geq E[X0], then X is a martingale. Hence the bivariate process is indeed a coupling with transition matrix p. Using Markov chains we construct a coupling to prove a theorem on the stochastic ordering of binomial distributions. Definition 16.1 A measure $\mu \in M1$ (R) is called infinitely divisible if, for every $n \in N$, there is a $\mu n \in M1$ (R) such that $\mu * n = \mu$., th }, we have $P\mu \circ XJ - 1 = \mu \otimes n - 1 k = 0 \kappa tk + 1 - tk$. (ii) (Monotonicity) If $X \ge Y$ a.s., then E[X |F] $\ge E[Y |F]$. 432 17 Markov Chains The following theorem was shown by Strassen [161] in larger generality for integral orders. In this case, we write $X \sim N\mu, C$. Hence, by Theorem 17.11, X is a Markov chain with transition matrix p. (i) (Xi) i \in I is exchangeable. HypB1,...,Bk;n {(b1, . 2 - 1 Step 6. Then H n \in E, and we have Htn (ω) \rightarrow Ht (ω) for all t > 0 and $\omega \in \Omega$. 11.1 Doob's Inequality. (i) Show that the distribution of (Xn) $n \in Z$ is uniquely determined by the values mn := E[X1 · X2 · · · Xn], $n \in N$. for some $r \in N$. Is this random walk recurrent or transient? & Exercise 13.1.6 Let μ be a Radon measure on Rd and let $A \in B(Rd)$ be a μ -null set. 9.1 Processes, Filtrations, Stopping Times 215 Example 9.8 (i) The Poisson process with intensity θ and the random walk on Z are processes with stationary independent increments. , An to occur jointly vanishes as $n \to \infty$. & Exercise 8.2.5 Show the conditional Markov inequality: For monotone increasing $f:[0,\infty) \to [0,\infty)$ and $\varepsilon > 0$ with $f(\varepsilon) > 0$, *) * E $f(|X|) \neq \varepsilon$ [$F \leq .$ We start this section by presenting as the main result Prohorov's theorem [136]. The other claims are evident. Now let $s \in K$ and $n \geq n0$. Thus, from (10.6), we recover the statements of Theorem 10.4 and Example 10.8. Later we will derive a formula similar to (10.6) for stochastic processes in continuous time (see Sect. Independence of random variables can be characterised). Takeaways A random variable X is called infinitely divisible if for any $n \in N$ it can be written as a sum of n independent and identically distributed random variables. 5.1 Rolling a die n times: Probabilities for Sn /n. In particular, pk (x, y) > 0. 216 6 i=1 i=1 (ii) Consider now the events A1 := { $\omega \in \Omega : \omega 1 = \omega 2$ }, A2 := { $\omega \in \Omega : \omega 1 = \omega 3$ }, A3 := { $\omega \in \Omega : \omega 1 = \omega 3$ } A = 17.5 Application: Recurrence and Transience of Random Walks In this section, we study recurrence and transience of random walks on the Ddimensional integer lattice ZD, D = 1, 2, . Corollary 16.8 Let $\phi : \mathbb{R} \to \mathbb{C}$ be continuous at 0. (iii) Compute mXY $\delta \to mX$ for $\delta \downarrow 0$. We will see that every signed measure has such a representation. Thus the Morse code can be interpreted as a ternary prefix code. Takeaways A coupling is a probability measure on a product space with given marginals. Summing over the connected components Z of HL with at least one point in TL, we obtain $\# u \in HL$: degHL (u) = 1 $\geq \#TL$. Hence we will consider different methods of proof that yield further insight into the problem. Let C C Rd be bounded, convex and open with 0 \in C. If X and Y are independent, then $\phi X + Y = \phi X \cdot \phi Y$. If the Markov chain X with weights C is recurrent, then the Markov chain X with weights C is recurrent. 24. Theorem 15.11 (Discrete Fourier inversion formula) Let $\mu \in$ Mf (Zd) with characteristic function $\phi \mu$. Proof First note that $U\phi \subset E1$ is Borel measure by Exercise 1.1.3. Hence the conditions make sense. Proof Let x, $y \in E$, x = y, be such that F(x, y) > 0. For the sake of distinction, we sometimes call λ the Lebesgue-Borel measure and $\lambda *$ the Lebesgue-Borel measure. i=1 We aim at extending μ to a measure on σ (A). Lemma 20.15 Let $p \ge 1$ and let X0, X1, As the family (Pn) $\in N$ is tight, by Theorem 15.22, (ϕ n) $\in N$ is uniformly equicontinuous. Then, for all $n \in N$, $* 0.8 \gamma$ sup P Sn* $\leq x - \Phi(x) \leq 3 \sqrt{1 + 1}$. For $\epsilon > 0$, let $A\epsilon = \{x \in I : D + \phi(x) \geq \epsilon + \lim_{n \to \infty} 1 + \varphi(x) \leq \epsilon + \lim_{n \to \infty} 1 + \varphi(x) \geq \epsilon + \lim_{n \to \infty} 1 + \varphi(x) \leq \epsilon + \lim_{n
\to \infty} 1 + \varphi(x) \leq \epsilon + \lim_{n \to \infty} 1 + \varphi(x) = 0$ Moments . Since we have $X \ge Y$, we get $E[1A(X - Y)] \ge 0$ and thus P[A] = 0., Xn be i.i.d. random variables with distribution μ . choose $\varepsilon(t) > 0$ and $C(t) < \infty$ such that $|f(r) - f(s)| \le C(t) \cdot |r - s|y$ for all $r, s \in Ut := U\varepsilon(t)(t)$. $n \to \infty$ (i) Show that $E[X\tau m | F\sigma n] \to E[X\tau m | F\sigma n]$ $(Xn,i) \in N0$, $i \in N$ be i.i.d. random variables with P[X1,1 = k] = pk for $k \in N0$. (ii) If $A = 2\Omega$ or $A = \{\emptyset, \Omega\}$, then any map $X : \Omega \to \Omega$ is A - A-measurable. 406 17 Markov Chains If we let rs = ps - ps, then $rs \infty \le 2$ and $q\infty \le 2\lambda$; hence sup $rs \infty \le sup s \le t s 0$ ru ∞ du $\le 2\lambda t$ sup $rs \infty$. as in (5.13) does then yield P[A]From Kolmogorov's inequality, we derive the following sharpening of the strong law of large numbers. $j \in J j \in J$ Remark 14.5 The concept of the product topology: If ($(\Omega i, \tau i), i \in I$) are topological spaces, then the product topology with respect to which all coordinate maps $i \in I Xi : \Omega \to \Omega$ are continuous. We say that (un) $n \in N$ converges vaguely to μ , $n \to \infty$ formally $\mu \to \mu$ (vaguely) or $\mu = v$ -lim μn , if $n \to \infty$ f dun $- \to f$ du for any $f \in Cc$ (E), $\omega n \in E$ be the atoms of ΞN and let N1, ..., ek $\in E$ be the atoms of ΞN and let N1, ..., ek \in E be the atoms of ΞN and let N1, ..., ek $\in E$ be the atoms of ΞN and let N1, ..., ek \in E be the atoms of ΞN and let N1, ..., ek \in E be the atoms of ΞN and let N1, ..., ek \in E be the atoms of ΞN and ..., ek $\in E$ be the atoms of ΞN and ..., ek $\in E$ be the atoms of ΞN and ..., ek \in E be the atoms of ΞN and ..., ek \in E be the atoms of ΞN and ..., ek \in E be the atoms of ΞN and ..., ek \in E be the atoms of ΞN and ..., ek \in E be the atoms of ΞN atoms at the expected of ΞN at the expected of ΞN atoms at the expected of ΞN at the defined only up to null sets. $n \to \infty - 1/2 - n$ n for any $n \in N$, and (iii) Choose anf $\in C([0, 1])$ with f (2) = (-1) Cn 3 does not convergence in total variation coincide.) Further, let En := $\mu \in M1(\Sigma)$: $n\mu(\{x\}) \in N0$ for every $x \in \Sigma$ be the range of the random variables $\xi n(X)$ (ii) If in addition X is adapted to the filtration F, then for any $t \ge 0$, the map $\Omega \times [0, t] \rightarrow E$, (ω , s) $\rightarrow Xs$ (ω) is Ft \otimes B([0, t]) - B(E) measurable. For measures on (E, E), we introduce the following notions of regularity. The next goal is to characterize σ -subadditivity by a certain continuity property (Theorem 1.36). 18.4 Speed of Convergence 455 Case 2: N even. 1 Furthermore, $\#(Ai \cap Aj) = 6$ if i = j; hence $P[Ai \cap Aj] = 36$. Assume (iii). For any $i \in I$, let (Ωi , Ai) be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let $Xi : \Omega \rightarrow \Omega i$ be a measurable space and let XN-1) $\in \cdot |X0 = x, XN = y| = PU x, y, n$ for all $x, y \in Zd$ with pN(x, y) > 0 and for all $n \in N0$. Thus, without loss of generality assume that all measures are in $M \le 1$ (E). We have to show that there exists a vaguely convergent subsequence. In particular, Rd , Zd , RN , (C([0, 1]), $\cdot \infty$) and so forth are Polish. Then -k E[Yk(t, h, X)] = k! h k - 1) * hl $\phi(t + x) = 0$ h) - $\phi(t)$ - E eit X (iX)l l! . n \in N k=n Thus ν 0 μ ., Xn /n). Lp -ergodic As a consequence, we obtain the statistical ergodic theorem, but was published only later in [122]. 24 1 Basic Measure Theory (iii) (π -system) Let A, B \in M(μ *) and E \in 2 Ω ., $\theta kN-1$. Example 17.5 Let Y1, Y2, The convolution of the transition probabilities translates into powers of the characteristic function; hence φ n (t) = ei)t,x* pn (0, x). \blacklozenge As in Example 17.6, we will construct a Markov process for a more general Markov process for a more general Markov semigroup of stochastic kernels. Hence, for i $\in \Lambda$ and $\sigma \in \{-1, +1\}$, $\pi(x i, \sigma x-i) = \pi(x i, \sigma)$ $\pi(\{x, j, -1, x, j, +1\}) = -\beta H(x) + e -\beta$ The same calculation $- \le t - 2$ Var[Sn]. Hence, it is enough to show that $\mu 1$ (K) = $\mu 2$ (K) for any compact set K. By induction, we get P1 [Xt > n] = fn (t) = (1 - e - t)n for all $n \in N$, $t \ge 0$. In this case, (Ω , A, P, τ) is called a measure-preserving
dynamical system. \bullet Example 12.29 (Pólya's urn model) (See Example 14.41, compare also [17, 135] and [58].) Consider an urn with a total of N balls among which M are black and M - N are white. Equip (as in Theorem 1.64) the probability space $\Omega = E$ N with the σ -algebra A = σ ({[$\omega 1$, . Hence we assume (1 - p1)n1 = (1 - p2)n2 . (ii) Let r, s > 0 and let β r, s be the distribution on [0, 1] with density $x \rightarrow \Gamma$ (r + s) r-1 x (1 - x)s-1 . 21.1 Continuous Versions .. We have to show that M(µ*) $\supset \sigma$ (A). < tn from I, and letting J := {t0, . n=1 Define Bn := En \cap A \in A. It shares the main properties of ordinary expectations (linearity, triangle inequality, monotone and dominated convergence, Jensen's inequality) and in addition has the so-called tower property. Let X be an Rd -valued random variable such that $1\% \& -1/2 P[X \le x] = det(2\pi \Sigma) exp - t - \mu, \Sigma - 1 (t - \mu) \lambda d (dt) 2 (-\infty,x]$ for $x \in Rd$ (where) · , · * denotes the inner product in Rd). 1.4 Measurable Maps . * 8.3 Regular Conditional Distribution Let X be a random variable with values in a measurable space (E, E). Klenke, Probability Theory, Universitext, 273 274 13 Convergence of Measures 13.1 A Topology Primer Excursively, we present some definitions and facts from point set topology. If f is integrable, then we 3 3 can define the integral f d μ := f d μ ., Xjn)]. + Xn for every n \in N. (ii) For a random walk started at a, show that the $\sqrt{\text{probability Pa}[\tau z < \tau a]}$ of visiting z before returning to a is Pa [$\tau z < \tau a$] = 1/3. The next corollary shows that the conditional expectation is in fact this minimizer. However, the first person is absent-minded and takes a seat at random. For example, for the Gamma distribution, we get $\alpha = 0$ and $n\Gamma\theta_1/n$ (A) = $\theta 1/n \Gamma(1/n)/n$ $n \rightarrow \infty x (1/n) - 1 e^{-\theta x} dx$. For every lk 3 3 k=0 ∞ k \in N, we have $E[Xk] = Ik x \nu(dx)$; hence $k=1 E[Xk] = (0,1) x \nu(dx) < \infty$. By Theorem 16.6, we 372 16 Infinitely Divisible Distributions $n \rightarrow \infty$ have $en(\varphi n - 1) \rightarrow \varphi$. We follow the proof in [39, Section 2.4]. Since fn = fn + f1 a.e., Theorem 4.9(iii) implies fn $d\mu = f + f1$ a.e. and f = f + f1 a.e. and f = f + fthus Y is also a G-backwards martingale (see Remark 9.29). 18.4 Speed of Convergence . (ii) If r = 12, then X is null recurrent., $\omega n] : (1 - p)\delta 0 + p \delta 1 \omega 1$, . \blacklozenge In order to formulate Fubini's theorem rigorously, we need the following definition. Sometimes q is also called the generator of the semigroup (pt) $t \ge 0$. If Fx 1, x 2, x 3 occurs, then we can find a point $v \in BL$ that is the starting point of three mutually disjoint (not necessarily open) paths $\pi 1$, $\pi 2$ and $\pi 3$ that end at x 1, x 2 and x 3. 404 17 Markov Chains 17.3 Discrete Markov Processes in Continuous Time Let E be countable and let (Xt) t $\in [0,\infty)$ be a Markov process on E with transition probabilities pt (x, y) = Px [Xt = y] (for x, y \in E). in . That is, $\lim \text{supn} \to \infty \text{ Xn} = -\infty \text{ and } \lim \text{infn} \to \infty \text{ Xn} = -\infty \text{ a.s.}$ We now consider the situation where the sequence $w = (wi -)i \in \mathbb{Z}$ is also random. The general case can be inferred inductively. $n \to \infty$ (ii) For any $\lambda \in \mathbb{R}^d$, there is a random variable X λ such that λ , Xn * $\Rightarrow X\lambda$. For which values of p and q do we db . If the values of p and q do we db . The values of p and q do we db . sequences whose first n values are $\omega 1$, At the points $x 0 \in A$, we apply the voltages u(x 0) (e.g., using batteries). $\bar{} \bullet$ Exercise 18.2.3 Let X be an arbitrary aperiodic irreducible recurrent random walk on Zd. (iii) For $\tau + s$, this is a consequence of (9.18) (with the stopping time $\sigma \equiv s$). Theorem 8.37 (Regular conditional distribution) Let $F \subset A$ be a sub- σ algebra. be independent random variables (and independent of Y1, Y2, . (ii) If A, B \in D with A \subset B, then I1B\A = I1B - I1A is measurable, where we used the fact that κ is finite; hence B \ A \in D. We conclude q = 1. We construct a probability measure P on (Ω , A) such that the stochastic process X has independent, stationary, normally distributed increments (recall Definition 9.7). + Xn = $n1/\alpha$ X1. Assume that lim infh $\downarrow 0 \phi(x - h) \leq \phi(x) - \varepsilon$ for some $\varepsilon > 0.8$ Exercise 17.6.2 Let X = (Xt) t ≥ 0 be a Markov chain on E in continuous time with Q-matrix q. It is enough to show (Steps 1-3 below) that $\mu *$ is an outer measure (see Definition 1.46) and that (Step 4) the σ -algebra of $\mu *$ -measurable sets (see Definition 1.48 and Lemma 1.52) contains the closed sets and thus E. Hence in fact $v_{S} \perp u$. Probability generating functions are an important tool for the investigation of branching processes. and Finally, let ψ Zn be the p.g.f. of Zn. For these s, t, we thus have $|f(s) - f(t)| = \lim_{t \to \infty} |f_{t}(s) - f_{t}(t)| \le \varepsilon$. Assume in addition that I is a current flow (that is, it satisfies Ohm's rule with some potential u that is constant both on A0 and on A1). i=1 By Kolmogorov's 0-1 law (Theorem 2.37), the tail σ -algebra T is trivial; hence we have E[Z1 T] = E[Z1] almost surely. In both cases, determine the conditional distribution of Θ given R = r. Definition 2.44 The critical value pc for percolation is defined as pc = $inf\{p \in [0, 1]: \theta(p) > 0\} = sup\{p \in [0, 1]: \theta(p) = 0\} = inf\{p \in [0, 1]: \psi(p) = 1\} = sup\{p \in [0, 1]: \psi(p) = 0\}. i=1 Zn can be interpreted as the number of individuals in the nth generation of a randomly developing population. Intuitively, this fits well with our idea that the Y1, The so-called Wasserstein metric on M1 (E) is defined by 0 dW (P, Q) :=$ inf 1 (x, y) $\phi(d(x, y)): \phi \in K(P, Q)$. Hence $\mu^{\sim} R$ is a σ -finite, additive, σ -subadditive set function on the semiring Z with $\mu(\emptyset)^{\sim} = 0$. See, e.g., [147, Chapter III.7ff]. The function $z \to zr-1 \exp(-z)$ is holomorphic in the right complex plane. (ii) Define G := (I - p)-1., Xk) and such $k \to \infty$ that P[A Ak] $- \to 0$. By symmetry, we also get $\alpha m \le \alpha n$. Let us take a moment's thought and look back at how we derived the strong Markov property of Brownian motion in Sect. Clearly, 3 form on Cb (\bar{E}) by $f \rightarrow \mu(f) := f d\mu$. Monotone convergence (Theorem 4.20) now yields $f q \leq \bar{F} p < \infty$; hence $f \in Lq(\mu)$. Hence, in the recurrent case, the set grows sublinearly. 1 21.5 Construction via L2 -Approximation 539 by (21.24) (and Theorem 21.11), X is a Brownian motion. There are 37 pockets (in European roulettes), 18 of which are red, 18 are black and one is green (the zero). $n \rightarrow \infty$ (ii) The map s : RN $\rightarrow [0, \infty]$, $x \rightarrow \infty^{-1}$ the value i=1 |xi | is symmetric. Let (Xn,i)n,i \in N0 k=0 be an independent family of random variables with P[Xn,i = k] = pk for all i, k, n \in N0. Hence, for every $\omega \in B \setminus N$, we have $\phi E[X | F](\omega) = \psi \omega E[X | F](\omega) = \psi \omega E[X | F](\omega)$. Then A is a class of sets of the same type A is a class of the same type A is as A; however, on A instead of Ω . For the subspace $K \subset \infty$ of convergent sequences, $F: K \to R$, (an $n \in N \to lim$ an is a continuous linear functional. Hence ∞ An = n=1 ∞ c Acn \in A. We have shown that E1 $\subset \sigma$ (E4). fn is called the Bernstein polynomial of order n. 25.3 The Itô Formula ... (ii) Var[X] = 0 $\iff X = E[X]$ almost surely.*) (iii) The map $f: R \to R$ $R, x \rightarrow E (X - x)^2$ is minimal at x0 = E[X] with f (E[X]) = Var[X]. Hence, as the state space we get E = M1 (Σ), equipped with the metric of total variation d(μ, ν) = $\mu - \nu T V$. Then $X\tau = E[XT \ F\tau]$ and, in particular, $E[X\tau] = E[X0]$. (7.9) Remark 7.36 The definition of total continuity is similar to that of uniform integrability (see Theorem 6.24(iii)), at least for finite μ . Proof For $\omega^{\sim} 1$, define the embedding map i : $\Omega 2 \rightarrow \Omega 1 \times \Omega 2$ by i($\omega 2$) = ($\omega^{\sim} 1$, $\omega 2$). μx) * is an
invariant distribution. 15.1 for an example with n = 4. \blacklozenge Example 12.28 Let (Xn)n \in N be exchangeable and assume Xn $\in \{0, 1\}$. 2.1 Independence of Events We consider two events A and B as (stochastically) independent if the occurrence of A does not change the probability that B also occurs. Thus, for every $n \in N$, writing $Cn := E[X12n] = (2n)! 2n n! < \infty$, we have $(\sqrt{2n} = Cn | t - s| n \cdot 196 \ 8 \ Conditional Expectations Proof Uniqueness Let Y and Y be random variables that fulfill (i) and (ii). <math>n \rightarrow \infty n \rightarrow \infty$ $n \rightarrow \infty$ D Corollary 13.19 If $Xn \rightarrow X$ in probability, then $Xn \rightarrow \infty$., $Sn \} n k=1 Xk$ for 20.4 Application: Recurrence of Random Walks 503 denote the range of S; that is, the number of distinct points visited by S up to time n., $Xtn \in An$) * Px Xt0 $\in A0$, . Hence $E[\phi(X) -] \leq E[(aX + b) -] \leq |b| + |a| \cdot E[|X|] < \infty$. (17.27) It can be shown that (this is the Kantorovich-Rubinstein theorem [84]; see also [37, pages 420ff]) 0 dW (P, Q) = sup 1 f d(P - Q) : f \in Lip1 (E; R). \blacklozenge Theorem 1.60 The map $\mu \rightarrow F\mu$ is a bijection from the set of probability measures on R, B(R) to the set of probability distribution functions, respectively from the set of subprobability measures to the set of defective distribution functions. Then $C \subset B$ is compact and $\mu(B \setminus C) < \epsilon$. Consider a gamble in a casino where in each round the player's bet either gets doubled or lost. (In fact, we could use the theory of branching processes to show that pc = 12.) (iii) For $p \in (pc, 1)$, show that with positive probability there are at least two infinite connected components. Let $\lambda \in \mathbb{R}$ and $s \in \mathbb{R}$. fdd fdd $n \to \infty$ $n \to \infty$ fdd fdd Lemma 21.36 Pn $\rightarrow P$ and Pn $\rightarrow Q$ imply P = Q. \blacklozenge Theorem 12.10 Let $X = (Xn) \cap \{d(f, fn) > \epsilon\} \leq \delta + \epsilon - 1 d^{\sim} N$ (f, fn) $\rightarrow \delta$. $n \to \infty$ fdd fdd Lemma 21.36 Pn $\rightarrow A$ and $n \to \infty$ fdd fdd Lemma 21.36 Pn $\rightarrow \infty$ fdd fd everywhere, then 3 f dµ ≤ 3 g dµ., Xtn) is n-dimensional normally distributed. 340 15 Characteristic Functions and the Central Limit Theorem - * b - = b - (v) br, p r + s, p for r, s > 0 and p ∈ (0, 1]. Remark 1.17 The following three statements hold: (i) E ⊂ σ (E). Furthermore, a(x) - := lim sup an (x) defines a symmetric map RN \rightarrow R \cup { $-\infty$, $+\infty$ }. Fix ε > 0 and p ∈ (0, 1]. Remark 1.17 The following three statements hold: (i) E ⊂ σ (E). Furthermore, a(x) - := lim sup an (x) defines a symmetric map RN \rightarrow R \cup { $-\infty$, $+\infty$ }. 0 and choose as \in (a, b) such that F (as) - F (a) < $\epsilon/2$. \cup An) < ∞ . Proof Let $\mu 1$, $\mu 2 \in$ Mf (Rd) with $\phi \mu 1$ (t) = $\phi \mu 2$ (t) for all t \in Rd . If $f \ge 0$ or $f \in L1$ ($\mu 1 \otimes \mu 2$), then $\omega 1 \rightarrow f(\omega 1, \omega 2)$ $\mu 1$ (d $\omega 1$) is $\mu 2$ -a.e. defined and A2 -measurable, and $\Omega 1 \times \Omega 2$ f ($\mu 1 \otimes \mu 2$) = f ($\omega 1, \omega 2$) $\mu 2$ (d $\omega 2$) is $\mu 1$ -a.e. defined and A1 -measurable, (14.6) $\omega 2 \rightarrow f(\omega 1, \omega 2)$ $\mu 1$ (d $\omega 1$) is $\mu 2$ -a.e. defined and A2 -measurable, and $\Omega 1 \times \Omega 2$ f ($\mu 1 \otimes \mu 2$) = f ($\omega 1, \omega 2$) μ^2 (d ω^2) $\Omega^1 \quad \Omega^2 = f(\omega_1, \omega_2) \\ \mu^1$ (d ω^1) $\Omega^2 \\ \mu^1$ (d ω^1) $(14.7) \\ \mu^2$ (d ω^2). For every $M \in N$, we have $Px1[\tau An0 \le M] \le M$ $n \rightarrow \infty$ $Px1[Xk \in An0] - \rightarrow 0$. (2.1) Example 2.1 (Rolling a die twice) Consider the random experiment of rolling a die twice. Definition 13.9 Let $F \subset M(E)$ be a family of Radon measures. Hence $A \in Ai$ for any $i \in I$. We say that X is integrable if $E[X] \in M(E)$. Then PX =: Berp is called the Bernoulli distribution with parameter p; formally Berp = $(1 - p) \delta 0 + p \delta 1$. $\in Lp(\mu)$. 17.3 Discrete Markov Processes in Continuous Time 405 Proof Let I be the unit matrix on E. Theorem 1.19 (Dynkin's π - λ theorem) If $E \subset 2\Omega$ is a π -system, then $\sigma(E) = \delta(E)$. It remains to show that κ is a stochastic kernel. be independent random variables with distribution P., Xk) \in Ck } for all k \in N. 376 16 Infinitely Divisible Distributions Definition 16.16 A σ -finite measure ν on R is called a canonical measure ν on R is called a canonical measure if $\nu(\{0\}) = 0$ and 2 (16.10) x \wedge 1 $\nu(dx) < \infty$. If A \in I, then, for every n \in N, A = τ -n (A) = { $\omega : \tau n(\omega) \in A$ } $\in \sigma(Xn, Xn+1, .)$ The uniqueness of the decomposition is trivial. Then $0 \le 2n \le 2Y$ and $2n - \rightarrow 0$. Thus, again by Lemma 14.23, the map $\omega 0 \rightarrow \kappa 1 \otimes \kappa 2$ ($\omega 0$, A) = $\kappa 1 (\omega 0, \omega 1)$ is well-defined and A0 -measurable. 16.1 Lévy-Khinchin Formula For the sake of brevity, in this section, we use the shorthand "CFP" for "characteristic function of a probability of measure on R"., An \in Bb (E) are arbitrary, then there exist 2n - 1 pairwise disjoint sets B1, \cup FN $\in \sigma$ (A1 \cup . We come back to this in Example 3.4(iv). Clearly, the value of N does not change if we shift all edges simultaneously. The problem of finding the smallest number N such that any n dx, n \geq N can be written as a nonnegative integer linear combination of k1, $n \rightarrow \infty$ If $x \in R$, then '1(=1 P[X* \le x*] = lim P X* \le x* + n \rightarrow \infty n and '1(P[X* < x*] = lim P X* ≤ x* - = 0. As a shorthand, we say that a family (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is
independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distributed") if (Xi) i ∈ I is independent and identically distribu probability measures on Rd and define the kernels $\kappa i : (Rd, B(Rd)) \rightarrow (Rd, B(Rd)), i = 1, 2, by \kappa 1 (x, dy) = \mu(dy) and \kappa 2 (y, dz) = (\delta y * \nu)(dz).$ Then A3L, $0 \subset Fx 1, x 2, x 3$. For $A \in F$, $r \in Q$ and $B = (-\infty, r]$, by (8.16), $\kappa(\omega, B) P[d\omega] = A) *) * P Y \in B$ [F dP = P A $\cap \{Y \in B\}$. (ii) Let r > p > 0. 26.2 Weak Solutions and the Martingale Problem ... We enter the realm of probability theory exactly at this point, where we define independence of events and random Walk in a Random Environment (Compare [143, 175] and [76, 77, 96].) Consider a Markov chain X on Z that at each step makes a jump either to the left (with probability wi-) or to the right (with probability wi+) if X is at $i \in Z$. The invariant measures are the nonnegative linear combinations of the measures $\mu 1$ and $\mu 2$ ($\{x\}$) = 1 and ($\{x\}$) = 1 The metric d from (1.9) induces the product topology on Ω ; hence, as remarked in Example 1.40, (Ω , d) is a compact metric space. \sim Now x \rightarrow G(x, y) is harmonic on E \ A. In particular, does not have atoms at 0 or 1), almost surely we draw infinitely many balls of each color. $n \rightarrow \infty$ fdd Proof As in (21.44) for $0 \le t1 \le t2$, $\lambda 1$, $\lambda 2 \ge 0$ and $x \ge 0$, we get, + $('(^n - \lambda - \lambda Z^2 n + \lambda Z^2 n - \lambda Z^2 n + \lambda$ 1.96, there exists a sequence of simple functions (fn)n \in N with fn \uparrow f. Since μ is \emptyset -continuous and is thus a signed measure on A. Give an example of a measure on example on example color as the ball that we return., xk) = k fi (xi) for any $k \in N$. Hence the aim is to study the 3 φ asymptotics of $Z\epsilon$:= $e\varphi(x)/\epsilon \mu\epsilon$ (dx) as $\epsilon \to 0$. If in Theorem 20.20 the random variables Xn are not integer-valued, then there is no hope that Sn = 0 for any $n \in N$ with positive probability. Assume that $X^{\circ} 0 = 0$. 10]. However, this was the claim. For S, T > 0, by Lemma 21.5, two such modifications XS and XT are indistinguishable on $[0, S \land T]$; hence $\Omega S, T :=$ there is a t $\in [0, S \land T]$; hence $\Omega S, T :=$ there is a t $\in [0, S \land T]$; with XtT = XtS is a null set and thus also $\Omega \infty := \Omega S, T$ is a null set. 20.5 Mixing Ergodicity provides a weak notion of "independence" or "mixing". E[Xn]. distribution of the individual random variables. Now define κY , $F(\omega, \varphi(A))$ for $A \in E$. Via the connection with subordinators, in the third section, we construct two distributions that play prominent roles in population genetics: the Poisson-Dirichlet distribution and the GEM distribution. Ik, $n \rightarrow \infty$ Show that fn (x) $- \rightarrow f(x)$ for λ -almost all λ $\in [0, 1]$. Proof For k $\leq n$, we have Mn ($\tau(\omega)$) \geq Sk ($\tau(\omega)$). Proof (i) (ii) (iii) time. In this case, $t \rightarrow N t$ is monotone but it is not right continuous, although (i) hold. 1. Let (An)n $\in N$ be a sequence in A with $\alpha = \lim \phi(An)$. Let $F \in (Lp(\mu))$. By Weierstraß's approximation theorem (Example 5.15), there is \sqrt{a} sequence (pn)n $\in N$ to be a sequence (pn)n $\in N$ be a sequence in A with $\alpha = \lim \phi(An)$. Let $F \in (Lp(\mu))$. where the content was defined on an algebra in the first place, the measure on sets of the σ -algebra can be approximated arbitrarily well by sets from the algebra. 229 233 239 11 Martingale Convergence Theorems and Their Applications . Define $\phi + (A) := -\phi(A \cap \Omega -)$., n $\Xi \infty = (12.5)$ l=1 Drawing without the algebra can be approximated arbitrarily well by sets from the algebra. replacements thus asymptotically turns into drawing with replacements. $|n| \rightarrow \infty$ Hence τ is mixing. and ∞ 0 Fn = n=1 1 sup Sk > $\epsilon \cap F = F$, $k \in \mathbb{N}$ k * $n \rightarrow \infty$) *) hence Fn $\uparrow F$. However, for $d \ge 2$, the condition F1 \ge F2 is weaker than $\mu 1 \le st \mu 2$. $1 + (\delta/2) n \rightarrow \infty$ Var[Sn] lim (15.7) l=1 Lemma 15.42 The Lyapunov condition implies the Lindeberg condition. Furthermore, for every t \in I, By (i), we have P[N] Nt \cap R \subset (Nr \cap R) \subset N. (ii) I \subset R is a (possibly unbounded) interval and X and Y are almost surely right continuous. For any choice of sets Ai \in Ei, i \in N, the family (Ai)i \in N is independent; hence (Ei)i \in N is independent. Remark 5.2 (i) The definition in (ii) is sensible since, by virtue of Theorem 4.19, $X \in Ln$ (P) implies that Mk < ∞ for all k = 1, . Now that we have established (iii), by Exercise 6.1.4, we see that $\infty n=1$ (Yn - E[Yn]) converges almost surely., xn) with x0 = x and xn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. On the other hand, any predictable H has the form Hn = y. (i) In both cases, $\sqrt{determine the conditional distribution of Y given X} = x$. Fn (D1, Note that in the last step, we used the fact that p - p/q = 1. The rate for the decay can be computed via the Legendre transform of the logarithmic moment generating function., d - 1). It is enough to show that sup fn $d\mu \ge g d\mu$. 472 19 Markov Chains and Electrical Networks Definition 19.21 Let I be a flow on E \ A. Example 9.13Let I = No and let Y1, Y2, Let p be a probability vector on E 1 × E 2 with marginals p1 and p2. Now, if Kn $\uparrow \infty$, n $\rightarrow \infty$, then for every N > 0, we have + P sup U⁻ tKn, n $\Rightarrow 0$ in C([0, ∞)). By induction, we get rs = 0 hence p s ≥ 0 . In particular, independent square integrable random variables are uncorrelated. $j \in J j \in J / R$ (ii) Ai = $\sigma(Z) = \sigma Z E$, R. Hence E[Xm 1B] = C \in Zm : P[C] > 0 Q(C) P[C \cap B] = P[C] Q(C) = Q(B). & Exercise 3.3.2 Assume that we have a branching process Z = (Zn)n \in N0 with Z0 = 1 whose offspring distribution is given by $pk = 13 \cdot (2/3)k$, $k \in N0$., N}, then in the new generation it will be random and binomially distributed with parameters N and k/N. with PX = μ and PXn = μ for n $\rightarrow \infty$ every n \in N such that Xn $\rightarrow X$ almost surely. From an abstract point of view, the integral is monotone and linear and fulfills the triangle inequality, which allows to use it to define normed vector spaces of functions. Then + , + , Bk = P Aj = P[Aj] = 2014, Amazon.com, Inc. :(y, x) F Therefore, $\left[\mu x \left(\{y\} \right) = Ex \left[\tau x1 - 1 \quad n=0 \right] :(x, y) \right] = Ex \left[\tau (X\tau + t)t \in I \quad r=\kappa(X\tau, \cdot), \in A \text{ with An } T$ Ω and $\mu(An) < \infty$ for all $n \in N$. n n-1 P[A] = k=0 Thus $P[A] \in \{0, 1\}$; hence I is trivial and therefore τ is ergodic. It takes a little more work to show that there exists a countable set $G \subset X$ that is uniquely determined by X g, $g \in G$, and is an RCLL C0 (E) and a process X version of X. Hence, in (14.19) we can replace the normal distribution by any parameterized family of distributions (νt , $t \ge 0$) with the property $\nu t + s = \nu t * \nu s$. Takeaways Two events A and B are independent if P[A \cap B] = P[A] \cdot P[B]., μn on arbitrary (even different) measurable spaces. n=1 * (G n). left-hand side does not change if we change the order of the sets A1, A2, We formulate this principle as a theorem. (13.3) Arguing as above, this will show that $\mu 1$ (K) = $\mu 2$ (K) and will hence conclude the proof. (ii) Let Y ~ N0,1. $\leq mn \leq 0$, where mk := yakk - y - ak - 1 is the slope on the kth interval. (17.18) $-\alpha < \infty$ if and only if $\alpha > 1$. (i) (ii) (iii) (iv) μ n ($\{k\}$) μ n (A) ψ n (z) ψ (z) ψ (z) for all $k \in \mathbb{N}0$. Hence $\mu(A1) = \lim \mu(A1 \setminus An) = \mu(A1) - \lim \mu(A1$ $P[B] = 1 \ 1 \ 2 \ and \ P[A \cap B] = 4$. Let B1, B2, Show that $dP(\mu, \nu) = dP(\nu, \mu)$ for all $\mu, \nu \in
M1(E)$. < tn, PXt -Xs = N0,t -s for all t > s. In the special case I = N0, X is called a Markov chain. For any two points x, $y \in Z$, we thus have $P(x,y)[\tau^{\sim} < \infty] = Px-y[Zn = 0 \ for \ some \ n \in N0] = 1$. Then M is adapted to F and $E[Mn \ Fn-1] = Mn-1$ -E[Xn-1(In)Fn-1] + E[Xn-1(In+Nn)Fn-1] = Mn-1 - P[In = i]Xn-1(i) + P[In + Nn = i]Xn-1(i) + P[In + Nn = i] = P[In + Nn = i] =Define the linearly interpolated processes $1 Z^{-}$ tn := t - n-1 tn! Z nt n! + Z nt $< \infty$. By stationarity, $P[\sigma n < \infty] = 1$ for every $n \in N0$; hence P[B] = 1. ∞ Hence, let A1, A2, . This is defined by $dP(\mu, \nu)$:= $max\{dP(\mu, \nu), dP(\nu, \mu)\}$, (13.4) where $dP(\mu, \nu)$:= $max\{dP(\mu, \nu), dP(\nu, \mu)\}$, (13.4) where $dP(\mu, \nu)$:= $max\{dP(\mu, \nu), dP(\nu, \mu)\}$, (13.5) and where $B \in = \{x : d(x, B) < \varepsilon\}$; see, e.g., [14, Appendix III, Theorem 5]. Proof By Remark 21.10, X is characterized by (ii). Let P0 be a probability measure on ($\Omega 0$, A0). In this case, g d(f μ) = (gf) d μ . \blacklozenge Definition 9.12 (Predictable) Let I = N0 or I = N. \bigstar 40 1 Basic Measure Theory Theorem 1.88 (Measurability of continuous maps) Let (Ω , τ) be topological spaces and let f: $\Omega \rightarrow \Omega$ be a continuous map., n, the sequence (Pnk)n \in N of kth marginal distributions is tight. 18.3 State space decomposition of a Markov chain with period d = 3. Since E is complete, the limit $f(\omega) := \lim \rightarrow \infty$ fn (ω) exists. Hence we get (by Theorem 6.25) fn - f p = gn 1 - \rightarrow 0. An algebra on a finite set Ω is a σ -algebra. Theorem 13.11 Let (E, d) be a metric space. 21.9 Pathwise Convergence of Branching Processes 553 Remark: The distribution of M can be expressed by the Kolmogorov-Smirnov formula ([101] and [157]; see, e.g., [133]) $P[M > x] = 2 \infty 22 (-1)n-1 e-2n x$. 8.3 Regular Conditional Distribution . k=0 Hence $\tau An0 \uparrow \infty$ almost surely, and thus Fn $\downarrow \{\tau x 1 = \infty\}$ (up to a null set). Instead, we choose representatively one case. Now let $A \in G$. Let U be a random variable that is uniformly distributed on (0, 1). 18.3 for an illustration of the state space decomposition of a periodic Markov chain. Let ϕ : I \rightarrow R be continuous and in the interior I \circ twice continuous and in the interior of Y. define the first and second discrete derivatives of f: f(x) := f(x - 1) - f(x - 1) - f(x - 1) - 2f(x). Proof. Frequency 0.0072 0.0013 0.0004 0.0113 5.4 Speed of Convergence in the Strong LLN 135 Here '.' denotes a short signal while '-' denotes a long signal. Now find an example that shows that the conclusion of Kolmogorov's 0-1 law need not hold under this assumption. * Exercise 9.2.3 Show that the claim of Theorem 9.35 continues to hold if X is only a submartingale but if ϕ is in addition assumed to be monotone increasing. , Xk = xk] (1 - F (y, x)). The set of such cylinder sets is denoted by ZJ Takeaways The ergodic theorem yields a law of large numbers for the occupation times of a positive recurrent Markov chain. Hence W is bounded m-1 in L2 and thus in L1. However, the computations used in Sect. Proof Fix $\epsilon > 0$, and choose $\delta > 0$ such that $|fn(t) - fn(s)| < \epsilon$ for all $n \in N$ and all $s, t \in E$ with $d(s, t) < \delta$. Then: (i) (ii) (iii) (iv) (v) $|\phi X(t)| \le 1$ for all $t \in \text{Rd}$ and $\phi X(0) = 1$. The chain should be designed so that at each step, only a small number of transitions are possible 446 18 Convergence of Markov Chains in order to ensure that the procedure described in Example 17.19 works efficiently. Thus $A \setminus B = A \setminus (A \cap B) \in D$. 17.7 Stochastic Ordering and Coupling. Theorem 15.15 (i) Let $\mu 1$, $\mu 2$, (5.1) i, In particular, $Var[\alpha X] = \alpha 2 Var[X]$ and the Bienaymé formula holds, Var m i=1. Then ϕ is convex if and only if $\phi(x) \ge 0$ for all $x \in I \circ .$ (note that there are 4(2n + 1) edges that connect $(0 \leftrightarrow \infty) = \text{Reff} \circ n=0$ $1 = \infty$. Inductively, we get $Var[Wn] = \sigma 2 n+1$ $k=2 m-k \le \sigma 2 m < \infty$. First consider the case K = R. Basic number theory then yields that, for every $n \ge n := r \cdot x = 1$ (ki /dx), there are numbers m1, ... + Xn)/n $\rightarrow m$ in probability. Indeed, in this case, D := A2 \in A2 : $\omega 1 \rightarrow \kappa(\omega 1, A2)$ is A1 -measurable is a λ -system (exercise!). The gene frequencies k/N in this model can be described by a Markov chain X on E = {0, 1/N, Lemma 14.23 Let κ be a finite transition kernel from (Ω 1, A1) to (Ω 2, A2) and let $f: \Omega 1 \times \Omega 2 \rightarrow [0, \infty]$ be measurable with respect to A1 \otimes A2 - B([0, ∞]). Here we denoted I (x) := x \in A I (x). 15.1 Separating Classes of Functions 333 Proof By the dominated convergence theorem, $[-\pi,\pi)d = -i)t, x^* \phi \mu$ (t) dt = I – п.п.)а е – 1)t.; $\lim n \to \infty = \lim n \to \infty [-\pi,\pi]d = e^{-i}t, x^* \mu({y}) = i)t, y^* \mu({y}) dt |y| \le n$ we converge to the orem for measures, it can be shown that there is always a vector of (i) We always have $E[\phi(Xt)] < \infty$ for all $t \in I$. Using the approximation theorems for measures, it can be shown that there is always a vector of (i) We always have $E[\phi(Xt)] < \infty$ for all $t \in I$. Using the approximation theorems for measures, it can be shown that there is always a vector of (i) We always have $E[\phi(Xt)] < \infty$ for all $t \in I$. Using the approximation theorem for measures, it can be shown that there is always a vector of (i) We always have $E[\phi(Xt)] < \infty$ for all $t \in I$. countably generated σ -algebra $G \subset F$ such that for any $A \in F$, there is a $B \in G$ with $P[A \ B] = 0$. $\uparrow Z$ and , $n \in N$ is a consistent family. Then there is a $k \in N$ and states x1, Then, for $g \in Ef$, $F(g) = gf d\mu$., gxn. By the definition of En, the map $F : E \ N \to R$ is measurable, n-symmetric and bounded. Then τ is measure-preserving. Then we have $A \setminus Q$ An $\downarrow \emptyset$ and $n \rightarrow \infty \mu(A) - \mu(An) = \mu(A \setminus An) - 0$. Then, by Ohm's rule, the current flow along the ith wire is Ii = u(1)-u(0) = R1i. As μ is additive, we obtain $\mu(A \cup B) = \mu(A \cap B) + \mu(B \setminus A)$. \blacklozenge Takeaways The set of points visited by a random walk within the first n steps grows with n at a speed that is the probability of no return. • Reflection In the previous example one might be tempted to assume that the Yi are uncorrelated instead of independent. In order to get an aperiodic chain, for $\epsilon > 0$, define the transition matrix $p\epsilon := (1 - \epsilon)p + \epsilon I$, where I is the unit matrix on E. $n \rightarrow \infty$ (i) μ is called lower semicontinuous if $\mu(An) \rightarrow \mu(A)$ for any $A \in A$ and any sequence (An) $n \in N$ in A with An \uparrow A. \clubsuit Exercise 21.5.6 Let $t \in (0, 1)$ and f0 (x) := t as well as fn (x) := Show that $\infty 2 \sin(n\pi t) \cos(n\pi x) n\pi n = 0$ fn (x) for $n \in N$, $x \in [0, 1]$. Indeed, An := {n, n + 1, . By }x, y* = }y, x* \in K, denote an (undirected) edge that connects x with y., N - 2, (18.12) and $\lambda x N - 1 = rxN - 2$. We infer that * $n \rightarrow \infty$) n StnN) $\rightarrow L[(Bt1, ..., N + 1, ..., N + 2, ..., N$ that, for small $\varepsilon > 0$, the mass of $\mu \varepsilon$ is concentrated around the zeros of I. \blacklozenge Takeaways In order to compute characteristic functions, in many cases it is enough to have a table of characteristic functions, in many cases it is enough to have a table of characteristic functions for some repertoire of standard distributions and to know how characteristic functions. continuous, the kernel W := F - 1 ({0}) is a closed (proper) linear subspace of V., Mn, T, n are independent and Poi λ/n -distributed. show that f d(Cn ϕ n) n $\in N$. This is the so-called Doob decomposition. Macroscopically, the individual spins cannot be observed but the average magnetization can; that is, the modulus of the average of all spins, $mA(\beta) = x \in E - 1$ $\pi(x) x(i)$. Let kn = n and $Xn_i = \sqrt{Yn}$. This justifies the term reversible. emphasize the "time evolution" aspect rather than the formal notion of a family of random variables. By Exercise 19.1.1 (with $A = \{3, 5\}$, x = 2 and y = 3), the probability of visiting 3 before 5 is P = G(2, 3) = 13 29., $N, \varepsilon > 0$, be nonnegative numbers. Hence $\Omega = \{1, ..., k \}$ Exercise 4.1.2 (Sequence spaces) Now we do not assume $\mu(\Omega) < \infty$. For the first part of (iii), see, eg. 13.4. In order to formulate (and prove) this statement (de Finetti's theorem) rigorously in Sect. n=0 Hence there exists a nontrivial invariant measure μ (that is, μ ({0}) can be chosen ∞ strictly positive) if and only if n=0 pn = 0.2 \blacklozenge i=0 Example 19.27 Asymmetric simple random walk on E = Z with p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p \in (12, 1), p(x, x + 1) = p
\in (12, 1), p(x, x + 1) = p $(x, x + 1) = p \in$ (12, 1), p(x, x + 1) = p $(x, x + 1) = p \in$ (12, 1), p(x, x + 1) = p $(x, x + 1) = p \in$ (12, 1), p(x, x + 1) = p $(x, x + 1) = p \in$ (12, 1), p(x, x + 1) = p $(x, x + 1) = p \in$ (12, 1), p(x, x + 1) = p $(x, x + 1) = p \in$ (12, 1), p(x, x + 1) = p (x, x + 1) = p (x-1 = 1 - p is transient. $n \rightarrow \infty x \in R$ Proof Fix $x \in R$ and let Yn (x) = 1($-\infty, x$) (Xn) and Zn (x) = 1($-\infty, x$) (Xn) for $n \in N$., Yn) are determined. Then the following implications hold: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iii) \Rightarrow (iv) $(=\Rightarrow)$ (iii) \Rightarrow (iv) $(=\Rightarrow)$ (iii) \Rightarrow (iv) $(=\Rightarrow)$ (iv) (=>) (iv) (Chains We define for k, l = 0, Hence Zn /mn is an almost surely and L1 -convergent martingale. The smallest value $\mu * (E)$ that can be obtained by such an approximation is called the outer measure of E. Deletion of loops., $n \rightarrow \{1, ..., Ne \ convergent \ martingale.$ some analytic tools. However, we restrict ourselves to the case I \subset R. (8.2) i \in I Proof Due to the σ -additivity of P, we have P[A] = P i \in I. n $\rightarrow \infty$ k=1 Proof See, for example, [128]. \blacklozenge Example 13.28 (i) If E is compact, then M1 (E) and M \leq 1 (E) are tight. Let d be a complete metric on E. Consider an arbitrary w \in W \{0}. Lemma 20.7 (i) A measurable map $f: (\Omega, A) \rightarrow R$, B(R) is I-measurable if and only if $f \circ \tau = f$. (iv) Similarly, the vague topology τv on M(E) is the 3coarsest topology such that for all $f \in Cc$ (E), the map $M(E) \rightarrow R$, $\mu \rightarrow f d\mu$ is continuous. Theorem 12.17 Let $X = (Xn \ n \in N)$ X_{2} , (iv) Since $|Z_{t}| \leq |X_{t}| + |Y_{t}|$, we have $E[|Z_{t}|] < \infty$ for all $t \in I$. Then, for any $\varepsilon \in (0, 1)$, $n \to \infty$ $n \to$ E is Polish, then also the converse holds: F is tight = F is weakly relatively sequentially compact. Hence by the approximation theorem for measures (Theorem 1.65), there are mutually disjoint sets An = Vn \cap Cn \in A, n \in N, such that B \subset A := ∞ n=1 An and $\mu(A) \leq \mu(B) + \epsilon/2$. (iii) Give examples of sequences (px)x \in N such that the chain is (a) transient, (b) null recurrent, and (d) positive recurrent but ∞ n-1 exp - (1 - pk) n=0 k=0 = ∞ . Thus we consider sequence spaces p = Lp (N, 2N, μ). In order to obtain criteria for the convergence of Markov chains, we thus have to understand periodicity first. , Dn-1). 31 (i) Compute the expectation and variance of 0 Bs ds. • Takeaways Families of random variables are independent if the events they describe are independent (Definition 2.14). (17.28) Compare this representation of the total variation norm, 0 1 P - QT V = sup f d(P - Q) : f \in L^{\infty} (E) with $f \propto \leq 1$. We call $p = (p\omega)\omega\in\Omega$ the weight function of μ . By Theorem 21.38, it is thus enough to show tightness of $(Lx [Z n], n \in N)$ in M1 $(C([0, \infty))$. l=0 (19.9) (iii) (Parallel connection (see Fig.19.2)) Let $E = \{0, 1\}$. For $f, q \in L2(\mu)$, define $f, q^* := \}$, q^* , where $f \in f$ and $q \in q$. Since $\sigma n 2 \to \infty$, for every $\varepsilon > 0$ and for sufficiently large $n \in N$, we have $2K < \varepsilon \sigma n$; thus $n \to \infty |Xn,l| < \varepsilon$ for all l = 1, . Then i = 1 An $N \uparrow E$ for $N \to \infty$ for all $n \in N$. (iii) Let Ω be countably infinite and let $A := \{A \subset \Omega : \#A < \infty \text{ or } \#A <$ measurable. 1.3 The Measure Extension Theorem . We denote by τ^{-} the corresponding topology induced on R and by τ the usual topology induced on R and by τ the usual topology on R. Clearly, the events A1 , . Hence there is an $n \to \infty \to 0$. In this sense, the binary model is a complete market. R $(-K,K]d \propto \leq g \propto = g$ By assumption of the theorem, inequality, we conclude 3 g du1 = 3 g du2. Thus, by Slutzky's theorem, we n $\rightarrow \infty$ also have (Sn - Tn) \Rightarrow N0.1. Only the onedimensional marginal distributions are determined. Theorem 17.38 An irreducible discrete Markov chain is either recurrent or transient. Proof We check the conditions of Theorem 21.40. "(v) \Rightarrow (iv)" This is evident., K, where $Z := K j = 1 \exp(-\beta W j)$ is the normalising constant. We introduce the following notation. By the composition theorem 1.80), fy,n is measurable., XN = xN = N. A necessary condition for $M < \infty$ is of course that the series $\infty n=0$ (1 - pn) diverge; that is, that X is recurrent. H is then a gambling strategy. \bullet Example 1.72 Let $\mu = \delta \omega$ be the Dirac measure for the point $\omega \in \Omega$ on some measurable space (Ω , A). 17.3 Discrete Markov Processes in Continuous Time 409 For illustration, first consider the extreme situation where wn grows very quickly; for example, wn = 2n for every $n \in N$. Thus $\pi(VT) = v0 = E[VT]$. Define A = n=1 P[An |Fn-1] = ∞ and $A = \lim \text{supn} \rightarrow \infty$ An. Hence (since p 0 (x, z) = 0) $\mu x p(\{z\}) = \infty$ p n (x, z) = ∞ p n (x, z) = $\mu x (\{z\})$. We define the set of open edges as p E p := {e $\in E : Xe = 1$ }. Xn that cannot be extended to an infinite exchangeable family X1, X2, For any $x \in V$, there is a unique representation x = y + z where $y \in W$ and $z \in W \perp$. For all numbers $x, y \in R$, we have $(f(x) - f(y))(g(x) - g(y)) \ge 0$. $(12. \& Example 7.42 \text{ If } \mu + , \mu - \text{ are finite measures, then } \phi := \mu + -\mu - \in M \pm .$ Hence $\kappa 1 \otimes \kappa 2$ ($\omega 0, \cdot$) is σ -finite and is thus a transition kernel. $n \rightarrow \infty$ (ii) (μn) $n \in N$ is tight, and there is a separating family $C \subset Cb$ (E) such that $f d\mu = \lim n \rightarrow \infty f d\mu$ for all $f \in C$. For $I \in I$, let NI be the number of clicks after time a but no later than b. Remark 15.7 Let X and Y be independent nonnegative random variables with Laplace transforms LX := LPX and LY := LPX and LY := LPX and LY := LPX and Y be independent nonnegative random variables with values in a Borel space (E, E) (hence, for example, E Polish, E = Rd, $E = R\infty$, E = C([0, 1]), etc.). However, it fails for semirings. 288 13 Convergence of Measures Exercise 13.2.1 Recall dP from (13.5). Proof Apply Theorem 6.19 with the convex map H (x) = x p . 19.10). + Xn ~ bn,p and thus E[f (Sn /n)] = n f (k/n) P[Sn = k] = fn (p)., ωn] := { $\omega \in \Omega : \omega i = \omega i$ for any i = 1, In particular, if C(x, y) = 1 ($x, y \in K$), then X is called a simple random walk on (E, K)., kn is said to satisfy the Lindeberg condition if, for all $\varepsilon > 0$, Ln (ε) := kn ' ($12 \text{ n} \rightarrow \infty \text{ E Xn}$, l $12 \rightarrow 0$. < tn . \bullet Example 9.31 Consider the situation of the preceding example; however, now with E[Yt] = 1 and Xt = ts=1 Ys for t \in N0. (iii) ($\sigma - \cup$ -closedness) Let A, B \in D. Repeated application of Fubini's theorem and the translation invariance of λn yields $P[X + Y \le x] = P[(X, Y) \in A] \otimes 2 = 1A(u, v) fX(u) fY(v) \lambda n (dv) (-\infty, x] = Rn Rn Rn Rn (-\infty, x-v) fX(u - v) \lambda n (dv) fY(v) h (dv) fY$ $(fX * fY) d\lambda n$. Finally, let $g = \infty$. Example 17.24 (Poisson process) The Poisson process with rate $\alpha > 0$ (compare Sect. Then $\sigma \tau n \in N0$ h(P, $\tau) = h(P, \tau ; P)$. The following theorem is due to Lindeberg (1922, see [108]) for the implication (i) \Rightarrow (ii) and is attributed to Feller (1935 and 1937, see [51, 52]) for the converse implication (ii) \Rightarrow (ii) 13 in greater detail., c(ek) (ek) = c1 (el), . Conclude that if F is a filtration and if B is a Brownian motion that is an F-martingale. Assume X1, X2, . This will be helpful in many places. In this case, we write $x \leftarrow p y$. We define $S = -\infty$, 1, if -1 = D1 = D2 = . In this case, we write $x n \rightarrow \infty \Rightarrow X$ or $PXn \rightarrow PX$. However, the right hand sides are well-defined and by (16.1) are the exponentials of the left hand sides. (For finite σ -algebras F, this was shown in Example 7.39.) Indeed, let P = @ $\mu/\mu(\Omega)$ and Q = $\nu/\mu(\Omega)$. A more instructive approach is based on first constructing, independently of F, a sort of standard probability space on which we define a random variable with uniform distribution on (0, 1). Then XJI is measurable with respect to AI - AJ . Furthermore, by Lemma 15.49 we $3 n \rightarrow \infty n$ are independent as soon as we know X and record a success with probability X. i \in J by the induction hypothesis (2.8), we have $\mu(E_j) = \nu(E_j)$ for every $E_j \in E_j \cup \{\emptyset, \Omega\}$. A distribution on [0, 1] is uniquely characterized by its moments (see Theorem 15.4). 62 2 Independence Remark 2.15 (i) Clearly, the family (Xi)i \in I is
independent if and only if, for any finite set $J \subset I$ and any choice of $Aj \in Aj$, $j \in J$, we have $P'(\{Xj \in Aj \} = P[Xj \in Aj]$. Now assume that $(pt)t \ge 0$ are the transition probabilities of another Markov process X with the same generator q; that is, with lim $s \downarrow 0$ 1 p s (x, y) = q(x, y)., DT with values in $\{-1, +1\}$ and functions fn : Rn - 1 × $\{-1, +1\} \rightarrow R$ for n = 1, Therefore, $X0 \ge \max\{S1, ..., On \text{ the other hand}, n \rightarrow \infty \mu([n, \infty)) \rightarrow 0 \text{ and } \nu([n, \infty)) \rightarrow 0 \text{ and } \nu([n, \infty)) = \infty \text{ for any } n \in N. Xn, l > \varepsilon 2 \text{ Var}[Sn] l=1 (15.6) 358 15 \text{ Characteristic Functions and the Central Limit Theorem The array fulfills the Lyapunov condition if there exists a <math>\delta > 0$ such that kn) * 1 E $|Xn, l| 2 + \delta = 0$. By Corollary 2.22, the distribution P(W1, ..., Wk+l+1) has theorem The array fulfills the Lyapunov condition if there exists a $\delta > 0$ such that kn) * 1 E $|Xn, l| 2 + \delta = 0$. By Corollary 2.22, the distribution P(W1, ..., Wk+l+1) has theorem The array fulfills the Lyapunov condition if there exists a $\delta > 0$ such that kn) * 1 E $|Xn, l| 2 + \delta = 0$. By Corollary 2.22, the distribution P(W1, ..., Wk+l+1) has theorem The array fulfills the Lyapunov condition if there exists a $\delta > 0$ such that kn) * 1 E $|Xn, l| 2 + \delta = 0$. By Corollary 2.22, the distribution P(W1, ..., Wk+l+1) has theorem The array fulfills the Lyapunov condition if there exists a $\delta > 0$ such that kn) * 1 E $|Xn, l| 2 + \delta = 0$. density $x \rightarrow \alpha k + l + 1 e - \alpha Sk + l + 1 e - \alpha Sk + l + 1 (x)$, where Sn (x) := x1 + . (ii) It can be shown that Y is the (unique strong) solution of the stochastic (Itô-) differential equation (see Examples 26.11 and 26.31) dYt = 2 2Yt dWt , (21.48) where W is a Brownian motion. If in (4) we take open rectangles instead of open balls Br (x), we get B(Rn) = σ (E5). The moments can be read off from the derivatives at 0. Let vn, k(x) := (n - k)! n! n i1 ,..., ik $= 1 \# \{i1, ..., ik\} = k \delta(xi1, ..., xik)$ in the last but one equality and $B \in M(\mu *)$ in the last equality. $\sqrt{(i)}$ For the effective conductance between a and z, show that Ceff (a $\leftarrow \rightarrow z$) = 3. Show that lim supn $\rightarrow \infty$ Sn = ∞ almost surely. On the technical side, the conditional expectation is constructed via the Radon-Nikodym theorem. If the player bets on "red", she gets the stake back doubled if the ball lands in a red pocket. Case 1. $\in A$, $\phi \propto n=1$ An = $\propto \phi(An)$. The central limit theorem suggests that pn (0, 0) \approx CD n-D/2 as n $\rightarrow \infty$ for some constant CD that depends on the dimension D. The simplest choice would be (Ω , A) = R, B(R), X : R \rightarrow R the identity map and P the Lebesgue-Stieltjes measure with distribution function F (see Example 1.56). Let us change the perspective and ask: For fixed X, which are the martingales Y (with Y0 = 0) that can be obtained as discrete stochastic integrals of X with a suitable gambling strategy H = H (Y)? + 13.4 Application: A Fresh Look at de Finetti's Theorem (After an idea of Götz Kersting.) Let E be a Polish space and let X1, X2, . Denote by H y, t the set of maps [0, $1] \rightarrow R$ that are Hölder- γ -continuous at t and define H γ := t $\in [0,1)$ H γ , t. "(i) \Rightarrow (iii)" The equality follows by the individual ergodic theorem. For $x \in N0$, define the branching process X with x ancestors and offspring distribution q by Xn-1 X0 = x and Xn := i=1 Yn-1, i = 1 Thus Z is a Markov chain with transition probabilities p(i, j) = p * i (j), where p*i is the ith convolution power of p. We conclude 1 Ceff $(x1 \leftrightarrow \infty) = \lim Px1 [Tx1 = \infty] = pF(x1)$. For N, $\delta > 0$ and $\omega \in C([0, \infty))$, let V N $(\omega, \delta) := \sup |\omega(t) - \omega(s)| : |t - s| \le \delta$, s, $t \le N$. Let $\alpha := \nu(\{0\})^{\sim} u(1)$ and define $\nu \in M((0, \infty))$ by $\nu(dx) = u(1)(1 - e - x) - 1$ 1(0, ∞) (x) $\nu^{\sim}(dx) = e - tx 1 - u(t) := 0$ That is, u is the Laplace transform of ν^{\sim} which determines ν^{\sim} and ν uniquely. Proof We may assume x = 0. By Lemma 17.46, $(f(Xn))n \in N0$ are martingales; hence we have $[N1] < \infty$. That is, $\kappa(f)p = f q$. (P4) For any $I \in I$, we have $E[N1] < \infty$. That is, $\kappa(f)p = f q$. there exists a continuous map $f : R \to [0, \infty)$ such that $F (x) = xj1 - \infty dtn f (x) = xj1 - \infty dtn f (x) = c \in \{0, 1\}L$: ck = ck (e) for $k \leq l(e)$ the set of all dyadic sequences of length L that start like c(e). For $i \in N$, define $Ei = \{\omega \in \Omega : \omega i \in A\}$: $A \subset E$. Hence there is a $\gamma > 0$ such that $k |\phi(t)| > 1 - \gamma t + 2 e 2$ for all $t \in R$. As ϵ (Xn $n \in N$, define $Ei = \{\omega \in \Omega : \omega i \in A\}$: $A \subset E$. Hence there is a $\gamma > 0$ such that $k |\phi(t)| > 1 - \gamma t + 2 e 2$ for all $t \in R$. As ϵ (Xn $n \in N$, define $Ei = \{\omega \in \Omega : \omega i \in A\}$: $A \subset E$. is ergodic, 1An n \in N is also ergodic. κ is stochastic if Ki = 1 for all i $\in \Omega 1$. (ii) Show that the space (Cb ([0, ∞)), ∞) of bounded continuous functions, equipped with the supremum norm, is not separable. , k} = 0 lim sup n! n $\rightarrow \infty$ for all l $\in N$. Then (Ym)m \in N are independent geometric random variables with parameter p (see Example 1.105(iii)). Then Sn := X1 + . There exists a unique measure μ on σ (A) such that $\mu(A) = \mu(A)$ for all $A \in A$. Dominated convergence yields (1 ' $n \rightarrow \infty$ P A $\cap \tau - k$ (B) = E [Yn 1A] $- \rightarrow E$ [IA P[B]] = P[A] P[B]. Hence ϕ is strictly convex and ϕ (0) = E[X1] < 0. If X is measurable, we write X : (Ω, A) $\rightarrow (\Omega, A)$. Then Xi := Fi - 1 (U) := inf x \in R : Fi (x) \ge U is a real random variable with distribution μ (see proof of Theorem 1.104). By assumption, we have $\tau n < \infty$ almost surely for all n. Then σ (X) and σ (Y) are independent given F1 as well as given F3 but not given F2., Xk-1) A 268 12 Backwards Martingales and Exchangeability and * $n \rightarrow \infty$) An (fk) $-\rightarrow E$ fk (X1) A. The first condition of Theorem 21.40 is exactly (i). Thus F induces a continuous linear map F0: $V0 \rightarrow R$ by F0 (x + N) = F (x). $\alpha \alpha \alpha \mu(\{|f| > \alpha\}) \leq This$ implies $\mu(\{|f| > \alpha\}) = 0$ if $\alpha > F1$; hence $f \infty \leq F1 < \infty$. Then $\infty \geq E[\phi(X)|F] \geq \phi(E[X|F])$. I (x1) I (x0) Correspondingly, the effective conductance is Ceff (x0 \leftrightarrow x1) = Reff (x0 \leftrightarrow x1) -1. Since $\mu \perp \nu$, we get $\mu \cap (N \cap N) = \mu(n \in N \cap N) = 0$. Hence, for each N \in N, we have $\infty \infty 1 \leq$ N μ (AN) + μ AN \cap d f, fnkl > 2-l < ∞ . Chapter 20 Ergodic Theory Laws of large numbers, e.g., for i.i.d. random variables X1 , X2 , . For n \in N, we conclude inductively by Theorem 3.8 that ψ Zn + 1 = $\psi \circ \psi$ Zn = $\psi \circ \psi$ n = ψ n + 1 . Further, let (Ω , F, P) be a probability space and let I \subset R be arbitrary. (iii) Consider

the Poisson distributions Poia and Poiß for α , $\beta \ge 0$. Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2 j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ Then we can define a finite transition kernel from $\Omega 1$ to $\Omega 2$ by $\kappa(i, A) = j \in A$ Ki := $j \in \Omega 2$ to M 2 and M 2 to between these points. Further, for any $n \in N$, $e \in E$ pe δe let Xn : $\Omega \to E$, (ωm) $m \in N \to \omega n$, be the projection on the nth coordinate. (i) Give a characterization of A as in Exercise 1.1.4 (page 11). (ii) In the case $\alpha = 2$, we have: If PX is not concentrated at one point, then (16.32) implies that PX is in the domain of attraction of some distribution. Then, d formally, q = dt pt. By Theorem 1.81, we have $\sigma E = \sigma (\{E \cap A : E \in E\}) A = \sigma (\{X-1 (E) : E \in E\}) = \sigma (X-1 (E)) = \{A \cap B : B \in \sigma (E)\} = \sigma (E) A = \sigma (\{X-1 (E) : E \in E\}) = \sigma (X-1 (E)) = \{A \cap B : B \in \sigma (E)\} = \sigma (E) A = \sigma (\{X-1 (E) : E \in E\}) = \sigma (X-1 (E)) = \{A \cap B : B \in \sigma (E)\} = \sigma (E) A = \sigma (\{X-1 (E) : E \in E\}) = \sigma (X-1 (E)) = \{A \cap B : B \in \sigma (E)\} = \sigma (E) A = \sigma (\{X-1 (E) : E \in E\}) = \sigma (X-1 (E)) = \{A \cap B : B \in \sigma (E)\} = \sigma (E) A = \sigma (\{X-1 (E) : E \in E\}) = \sigma (X-1 (E)) = \{A \cap B : B \in \sigma (E)\} = \sigma (A \cap B) = \sigma (A \cap B$ $E[|h(X)|] < \infty$. Finally, consider (-f) to $n \rightarrow \infty$ obtain the reverse inequality lim inf f dun \geq f du. By construction, Q, q - < x < q + with F (q +) \leq F (x) + ϵ .). be square-free (that is, there is no number r = 2, 3, 4, . Clearly, for any q \in {p, p}, the family (Xe) $\in E$ of random variables is independent q p p (see Remark 2.15(iii)) and Xe ~ Berq. \clubsuit Exercise 18.4.6 Let $N \in N$ and let $E = \{0, 1\}N$ denote the N-dimensional hypercube. For every $\varepsilon > 0$, there exist events A $\varepsilon \Omega = E I$, $P = PX0 \varepsilon$ and P[B B $\varepsilon] < \varepsilon$. We define Tn := n Wk k=1 and interpret Wn as the waiting time between the (n - 1)th click and the nth click. Since (Ω, d) is separable (Theorem 21.30), every open set is a countable union of ε -balls. The proof presented here goes back to an idea of Dvoretzky, Erdös and Kakutani (see [40]). Then $+\phi(\text{Em}) = \phi \Omega$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) $-\rightarrow \phi(\Omega +) + \infty$ m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En} \setminus \text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En+1})$ (En \ En+1) + \infty m $\rightarrow \infty \phi(\text{En+1})$ (En \ En+1 superconductors along ∂B ., $ck \ge 0$ and c0 + . In particular, N0,1 is called the standard normal distribution. A set $A \subset U \in U \cup .$ Then d is said to be the period of that state. n i=1 For every $k \in An(\nu)$, we have (compare (23.14)) has a finite $U \subset U$ with $A \subset U \in U \cup .$ Then d is said to be the period of that state. n i=1 For every $k \in An(\nu)$, we have (compare (23.14)) $P[\xi n(X) = \nu] = #An(\nu) P[X1 = k1, . Indeed, here only the parameters of the Beta distribution change. Since we have not yet shown the existence of an independent family with this distribution, we content ourselves with Ye that assume only three values {p, p, 1}. Since j = 1 c c c U is closed and since U \cap K = \emptyset$, we get $\delta := d(U, K) > 0$. Since we do not have a general notion of an integral at hand at this point, for the time being we restrict ourselves to presenting the convolution formula for integer-valued random variables only. \neq Exercise 4.2.5 Let λ be the Lebesgue measure on R, $p \in [1, \infty)$ and let $f \in Lp(\lambda)$. m) * ∞ E Yn2 Lemma 5.20 \leq 4 E[[X1 |]. For example, it can be enough to check independence for intervals, rectangles or, in the discrete case, for single points. Define $Y_n := n1 \ k=0$ is uniformly integrable. A $\land w$ in for some $m \in \mathbb{R}$ if and only if $x \ P[|X| > x] \rightarrow 0$ and $E[X \ 1\{|X| \le x\}]$ $-\rightarrow$ m. First let f = 1A for A = A1 × A2 with A1 \in A1 and A2 \in A2. Hence, it will be important to find out which paths are P-almost surely negligible. After that, the chains run together. This shows (i). Hence Ω is separable and thus Polish. In order for x = (x0, . In the remainder of this section, assume that (E, C) is a finite electrical network. Then the following are equivalent. By the Portemanteau theorem (Theorem 13.16), $\mu = w$ -lim µnk; hence F is recognized as
weakly relatively sequentially compact. By F = σ (X), we denote the filtration generated by X. As f is continuous on the compact interval [0, 1], f is uniformly continuous. Now we extend the notion of independence from families of events to families of classes of events. Define $x + := \inf C \cap [0, \infty)$ as well as $x - := \sup C \cap (-\infty, 0]$. A for the arithmetic mean (Example 12.16), we can argue that limn $\rightarrow \infty$ An (ϕ) is T -measurable. We distinguish the cases where E[X] is in the interior I \circ or at the boundary ∂ I. Hence U is relatively compact; thus pK, ϵ has compact support and is thus in Cc (E) for all $\epsilon \in (0, \delta)$. For further reading, we recommend, e.g., [86, 118, 145, 152]. However, first we have to exclude the case where n is odd since here clearly pn (0, 0) = 0. (i) The family (Ai)i \in I is independent. Use the star-triangle transformation to remove the lower right node (left 19 513 54 in Fig. We present a general two-step construction principle that will be used in a similar form later in Chap. Hence, we define $\alpha := \alpha 1 > 0$ and get an $= n1/\alpha$ for all $n \in N$ (note that (16.21) implies a 1 = 1). Show that for any $\varepsilon > 0$, there exist finitely many pairwise disjoint sets U1, $\infty l=1$ Note that $|\log(1 + x) - x| \le x 2$ for |x| 0 and $pn = \lambda/n$, $n \in N$. Let $C \subset Cb$ (E; K) be an algebra that separates points. $\blacklozenge p \in Pn p \in P 58 2$ Independence If we roll a die infinitely often, what is the chance that the face shows a six infinitely often? (viii) Let $\theta > 0$ and let X be a nonnegative random variable such that $x P[X \le x] = P[X \in [0, x]] = \theta e - \theta t$ dt for $x \ge 0$. Using the statistic Mn, one can test if random variables of a known distribution are independent. Even if we can compute the speed of convergence (and in many cases, this is not trivial, we come back to this point in Sect. Let $Z = \lim n1$ Sn. 326 14 Probability Measures on Product Spaces If d = 1 and νt (($-\infty$, 0)) = 0 for all $t \in I$, then ν is called a nonnegative convolution semigroup. Then there does not exist a topology on the set of measurable maps $\Omega \rightarrow E$ that induces almost everywhere convergence. For every R the integral If (μ) := f d μ is well-defined and for every f \in Bb (E), If (μ) is well defined and finite. $n \rightarrow \infty l = 1$ Proof Let $mn := max \phi n, l(t) - 1$. sup $t \in [0, n] \cap Q Zt$ In the following, let $A := \sigma Xt$, $t \in [0, \infty)$., An }). By Sanov's theorem, ($\mu 0n$) $n \in N$ satisfies an LDP with rate n and rate function I (x) = H ($x|\lambda$), where H ($x|\lambda$) is the relative entropy of x with respect to λ . \blacklozenge Takeaways An irreducible Markov chain possesses an invariant distribution if and only if it is positive recurrent. A The main result of this chapter is Carathéodory's measure extension theorem. In particular, P U a, b < $\infty = 1$. Note that P[B] = 1. Hence $\Lambda * (z) = zt * - \Lambda(t *) = z \arctan(z) - \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(\arctan(z)) = 2t * - \Lambda(t *) = z \arctan(z) + \log \cosh(-1) + \log (-1) +$ We have to show that $\delta(E)$ is a σ -algebra. Proof (i) \Rightarrow (ii) Fix $\epsilon > 0$ and choose N \in N such that $\mu(\{N + 1, N + 2, . d\mu \text{ For example, the normal distribution } \nu = N0, 1 \text{ has the density } f(x) = with respect to the Lebesgue measure } \mu = \lambda$ on R. & Exercise 5.3.4 (Subadditivity of Entropy) For i = 1, 2, let E i be a finite set and pi a probability vector on E i. Hence $\tau \in i \in N$ Bi and thus $B \subset i \in N$ Bi and thus $B \subset i \in N$ bi and thus $B \subset i \in N$ bi and $\phi(2n-1)$ is continuous at 0 and $\phi(2n-1)$ (t) exists for all $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. (ii) For all $x, y \in E$, there exist $nx, y \in V$ and $Lx, y \in [0, -1)$ bi and $Lx, y \in [0, -1]$ bi and $E[X1] + 2\epsilon \lim \sup kn - 1 \operatorname{Skn} l \to \infty n \to \infty \leq 2\epsilon E[X1] \operatorname{almost surely.} n (23.12) \operatorname{However, by the central limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} ≥ log E n n \sqrt{-\tau c n 1 n \to \infty} + lim log N0, Var[X^1] ([0, c]) - lim n \to \infty n \to \infty n = 0. We have the contral limit theorem (Theorem 15.38), for every c > 0, (1 (' 1 ^ log E e - \tau Sn 1{S^n ≥ 0} > 0) > 0) = lim n \to \infty n \to \infty$ define E[X; A] := E[1A X]. $\bullet i=1$ In each of the three preceding examples, the effective resistance is a monotone function of the individual resistances. be an exchangeable sequence of random variables with values in E. Proof Let $F \subset E$ be countable and dense. Proof For $\epsilon > 0$, by Chebyshev's inequality, kn kn *) *) $2 n \rightarrow \infty P |Xn,l| > \epsilon \leq \epsilon - 2 E Xn,l$ $1 \{ |Xn, l| > \epsilon \} = \epsilon - 2 \ln(\epsilon) - \rightarrow 0$. Theorem 21.19 (Reflection principle for Brownian motion) For every a > 0 and T > 0, $\sqrt{} * 2 T 1 - a 2/2T P$ sup Bt : $t \in [0, T] > a = 2 P[BT > a] \leq \sqrt{}$. Theorem 7.7 Let $I \subset R$ be an interval with interior $I \circ and let \phi : I \rightarrow R$ be a convex map., n) is independent and Nti - Nti-1 $\sim Poi\alpha(ti - ti - 1)$. n = 1 It follows that, for every $n \in N$, $n \ge 1 > P[An] \le P|Xk2 - n - X(k-1)2 - n| \ge 2 - \gamma n \le C \ge -n(\beta - \alpha \gamma)$. The conductance of the wire that connects the points $x \in E$ and $y \in E \setminus \{x\}$ is denoted by $C(x, y) \in [0, \infty)$. Proof First consider the case where G is open. Let F be closed and let $\rho F_{,\varepsilon}$ be as in Lemma 13.10. Finally, let f, $g \in Lp(\mu)$. Lemma 15.29 If $\mu \in Mf(Rd)$ has characteristic function ϕ , then ϕ is positive semidefinite. If (Xi-1 (Ei))i \in I is independent, then (Xi)i \in I is independent, then the round is n at most C σ cos(π/N). i=0 Then Y = (Yn)n \in Z is a stationary process. \blacklozenge 7 Lp -Spaces and the Radon-Nikodym Theorem 182 Theorem 7.43 (Hahn's decomposition theorem) Let ϕ be a signed measure. 417.2 Discrete Markov Chains: Examples Let E be countable and I = N0. Then n=1 $\mu(A \cap E) = \infty$ $\nu(A \cap E) = \infty$ n=1 $\mu(A \cap E) = \infty$ $\nu(A \cap E) = \infty$ n=1 $\mu(A \cap E) = \dots$ n=1 $\mu(A \cap$ $e-2\pi in x = 2 \cos(2\pi n x)$ is Imeasurable but not a.s. constant. (ii) (Closedness under complements) Let $A \in D$., $Xn) \ge \phi(E[X1])$, We can use the jump times of X for an alternative derivation. but X0 = X1. This measure is called the restriction of μ to Ω . Lemma 1.42 (Uniqueness by an \cap -closed generator) Let (Ω, A, μ) be a σ -finite measure space and let $E \subset A$ be a π -system that generates A. For any $f \in Cb$ (E, ; R) and ; any $\epsilon > 0$, by the Stone-Weierstraß theorem, there exists a $g \in C$ with ; f - g; $\infty < \epsilon$. Hence ($|fn| - |fn\epsilon|$)+ 1 < ϵ for all $n \ge n\epsilon$. Define Xn = Xn, Θn and Yn = Yn, Θn and Yn = Yn such that $\mu(A1 \times \cdots \times An) = n \ \mu i$ (Ai) for Ai \in Ai, i = 1, . By the σ -subadditivity of β , $\mu * \infty$ Gn $\leq \beta$ n=1 $\infty \leq An$ n=1 Letting $\epsilon \downarrow 0$ yields $\mu *$ measure. $e \in E$ We first define a specific code and then show that it is almost optimal. Hence we have $u(t) = \lim un(t) = \lim$ $\Lambda(t) = \sup tz - = 2 2 t \in \mathbb{R} t \in \mathbb{R}$ Hence the rate function coincides with that of (23.4). Then define X = Z0 and 0 Yk = 1, 0, if $Zk \geq X$. (c) The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A. Example 1.77 (i) The identity map id : $\Omega \rightarrow \Omega$ is A - A-measurable. Let $r \in (0, 1)$ and $\tau r (x) = 1$. $x + r \pmod{1}$. 3(g) = F(g)
for $: g \rightarrow gf d\mu$ is in (Lp (μ)), and F Concluding, the map F is continuous and Ef \subset Lp (μ) is dense, we get F = F. 0.0 By Lemma 5.19, for $m \rightarrow \infty$, fm (x) = $m -2.1\{x, x\} \uparrow f(x) \le 4$ P[|X1| > x]. \bullet In particular, we have shown the following theorem., Xn) = $n-1 (X1 + .Define \lambda := -n1 \log(1 - p1) = -n2 \log(1 - p2)$. We conclude $A \in AI$. In the opposite case, v0 < v0, the strategy H - H 9.4 Discrete Martingale Representation Theorem and the CRR Model 227 ensures a risk-free profit. 19.7 that starts at x and at each step jumps to one of its neighbors at random with equal probability. There is an increasing sequence (Kn)n \in N of compact sets with Kn \uparrow Rd. If x is recurrent, then we conclude that $G(y, y) \ge \infty$ pl+n+k (y, y) $\ge pl$ (y, x)pk (x, y)G(x, x) = ∞ n=0 and hence also that y is recurrent. \bullet Example 18.7 Let E = Z and p(x, y) = 1/3 if $|x-y| \le 1$ and 0 otherwise. Here also the definition ends up with a product formula..., Xm , Y1 , . For the moment, note that the martingale $S = (1 - Sn)n \in N0$, just like the one in Example 9.31, has the structure of a product of independent random variables with expectation 1. 94 3 Generating Functions Case 1: $\lim_{x \to 1} \psi(z) \le 1$. (17.30) See [60] for a comparison of different metrics on M1 (E). the conditions of Theorem 1.36. & Exercise 13.2.6 (Lévy metric) For two probability distribution functions F and G on R, define the Lévy distance by $d(F, G) = \inf \varepsilon \ge 0$: $G(x - \varepsilon) - \varepsilon \le F(x) \le G(x + \varepsilon) + \varepsilon$ for all $x \in R$. We interpret Di as the result of a bet that gives a gain or loss of one 216.9 Martingales euro for every euro we put at stake. Hence, for n individuals, qk * n is the probability to have exactly k offspring. Since (19.5) implies equality, we infer f (x0) = f (y) for all $y \in SA(x0)$. Find an example that shows that this condition cannot be dropped. q With the aid of Kolmogorov's 0-1 law, we can infer the following theorem. (ii) By Theorem 13.34, it is enough to show that the sequence (Pn) $n \in N$ is tight., 1An (Y1)). It is of a certain theoretical interest and will be needed later that triangle functions are characteristic functions. Then Define A := lim $n \rightarrow \infty \infty$ k=n $(Ak) \leq lim n \rightarrow \infty \infty$ k=n 2-k = 0. Proof μ is σ -finite since $\mu \in M(E)$. 23.2 Large Deviations Principle . A state is recurrent if and only if the expected number of visits (Green function) is infinite. i=1 Therefore, lim sup $n \rightarrow \infty$ f dµn \leq Replacing f by (1 - f), we get (13.8). Then the following inclusion formulas hold: $\mu(A1 \cup . \blacklozenge f(x, \lambda) d\lambda n 162 6 Convergence Theorems Takeaways Consider a function of two variables that is$ continuous or differentiable with respect to one variable. From Exercise 15.1.3, we know that D Sn+1 = max{Z1, . One possibility is to construct n + 1 independent random variables Z0, . i~j The Boltzmann distribution π on E := {-1, 1}A for the inverse temperature $\beta \ge 0$ is defined by $\pi(x) = Z\beta - 1 \exp(-\beta H(x))$, where the partition sum $Z\beta = 1$ $exp(-\beta H(x))$ is the normalising constant such $x \in E$ that π is a probability measure. Now we compute $(f \circ \tau r)(x) = \infty$ cn $e2\pi in r$ for all $n \in Z$. p Proof For $p \in [1, \infty)$, use Jensen's inequality with $\phi(x) = |x|p$. 19.6.) paths follow different (x) is the normalising constant such $x \in E$ that π is a probability measure. Now we compute $(f \circ \tau r)(x) = \infty$ cn $e2\pi in r$ for all $n \in Z$. p Proof For $p \in [1, \infty)$, use Jensen's inequality with $\phi(x) = |x|p$. 19.6.) paths follow different (x) is the normalising constant such $x \in E$ that π is a probability measure. Now we compute $(f \circ \tau r)(x) = \infty$ cn $e2\pi in r$ for all $n \in Z$. p Proof For $p \in [1, \infty)$, use Jensen's inequality with $\phi(x) = |x|p$. directions for the first time, they will not have any common edge again, though some of the nodes can be visited by both paths. Let $\tau := \inf\{t \ge 0 : Bt \ge a\} \land 1$. Separability in metrizable spaces is equivalent to the existence of acountable base of the topology; that is, a countable set $U \subset \tau$ with $A = U \in U : U \subset A U$ for all $A \in \tau$. In fact, Kirchhoff's rule says that I is divergence-free on $E \setminus A$. Further, let $\phi : G \rightarrow R$ be convex. A map) \cdot , $\cdot * : V \times V \rightarrow R$ is called an inner product if: (i) (Linearity))x, $\alpha y + z^* = \alpha$)x, $y^* + y$, z^* for all x, y, $z \in V$ and $\alpha \in R$. $\in A \propto \infty$ such that $A \subset An$ and $\mu An \setminus A < \varepsilon$. -1, we get $w(i+l+1)/n - w(i+l)/n - wt + w(i+l)/n - wt + w(i+l)/n - wt \le 2c (k+1)\gamma n - \gamma$. For $x \in R$, define the left-sided limit 14.3 Kolmogorov's Extension Theorem 317 F (x-) = supy s, is consistent. 1 By Lemma 5.18, this implies the claim of Theorem 5.17. Then A, B are independent \Rightarrow P[A|B] = P[A] \Rightarrow P[B]. Suppose the claim of Theorem 5.17. Then A, B are independent \Rightarrow P[A|B] = P[A] \Rightarrow P[B]. $n \in N$ and let X1, Construct a countable algebra $C \subset Cb$ (R) that separates points. Then $\mu(E) \leq \lim \inf \mu(E)$. to, Some of the most successful and established books in the series have evolved through several editions, always following the evolution of (X1) and (E). . As in the proof of the martingale convergence theorem (Theorem 11.4), we infer the following. ♦ Remark 9.28 If we do not explicitly mention the filtration F, we tacitly assume that F is generated by X; that is, Ft = σ (Xs, s ≤ t). ♦ Example 17.28 (A variant of Pólya's urn model) Consider a variant of Pólya's urn model with black and red balls (compare Example 12.29). Further, let Sk = X1 + . (i) F is continuous and linear. The statements (iv) and (v) are immediate consequences of Lemma 16.25. (x) Let $\Omega = \{1, 2, 3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 2, 3, 4\}$, and $A = \emptyset$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 2, 3, 4\}$ and $A = \emptyset$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 2, 3, 4\}$ and $A = \emptyset$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 2, 3, 4\}$ and $A = \emptyset$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, $\{3, 4\}$, $\{2, 3\}$, $\{3, 4\}$, entropy H (P) of a probability distribution P (see Definition 5.25) measures the amount of randomness in this distribution. An n \in N n Exercise 20.6.3 Let pi be the transition matrix of a Markov chain on the vicinity of 3.5 (Fig. 15.1 Separating Classes of Functions 331 With the substitution $y = \log(x) - n$, we get (note that $\sin(2\pi(y + n)) = \sin(2\pi y) \propto 2 + \frac{1}{2} \sin(2\pi y) = 0$, where the last equality holds since the integrand is an odd function. be measurable maps $\Omega \rightarrow E$. (ii) (Linearity) Let $c \in R$. 8.2 Conditional Expectations 199 Proof First assume that $E[E[X | F]_2] < \infty$. 538 21 Brownian Motion Then (bn,k) is an orthonormal system:)bm,k, bn,l * = 1{(m,k)=(n,l)}. On the other hand, $\psi(t) = 12 + 12 \varphi(2t)$. Let $A \in B(Rd)$ and $\varepsilon > 0$. + Yn for $n \in N0$; hence G(0, 0) = n=0 P[Sn = 0]. Show that for any $A \in E$ and any $n \in N$ there exist pairwise disjoint sets A1, . . , n}. To this end it is enough to show that for any $n \in N$, the map $\omega \to
Yn(\omega) := dn(\omega 0, \omega)$ (see (21.28)) is A-measurable. The Metropolis algorithm for this chain accepts the proposal of the reference chain with probability 1 if $\pi(x_i) \ge \pi(x)$. The Haar functions bn,k are one such choice: Let $b0,1 \equiv 1$ and for $n \in N$. and k = 1, Definition 13.21 Let F, F1, F2, P $|X| \ge \epsilon \le f(\epsilon)$ + In the special case f(x) = x 2, we get P $|X| \ge \epsilon \le \epsilon - 2 E X 2$. "(ii) \Rightarrow (iv)" Convergence of the total masses follows by using the test function $1 \in Lip(E; [0, 1])$. For any $A \in A$, by (7.8) and (7.7) with $h = 1A \setminus E$, 1 - g f $d\mu = A$ Hence $f = A \cap E$ c g $d(\mu + \nu) = \nu(A \setminus E) = \nu a$ (A). If $\mu \in M(E)$, then we define $f d\mu := \operatorname{Re}(f) d\mu + i \operatorname{Im}(f) d\mu$ if both integrals exist and are finite. Assume that K > 0 is large enough such that f(x) = 0 for $x \in (-K/2, K/2)d$ and such d d that $\mu i R \setminus (-K, K) < \varepsilon$, i = 1, 2. Hence there are sequences (fn) $n \in N$ and (gn) $n \in N$ in $E + \operatorname{such} that fn \uparrow f$ and $gn \uparrow g$. \bigstar L1 Remark 6.10 If $fn \to g$, then f = g almost everywhere. Assume the conditions of Theorem 12.24 are in force. Corollary 13.30 Let E be a compact metric space. Then: (i) ϕ is continuous on I \circ and hence measurable with respect to B(I). We want to estimate the probability that there exists a point $x \in C p(0)$ with distance n from the origin. A family (Xi)i \in I of random variables with values in E is called exchangeable if *) *) L X(i) i \in I = L (Xi) i \in I = L (Xi) i \in I = L (Xi) i \in I for any finite permutation : I \rightarrow I. In other words, H must be predictable. Hence the intersection is nonempty. 2 k! Hint: Expand the bracket expression, sort the terms by the different mixed moments and compute by combinatorial means the number of each type of summand. The value of the portfolio, which is the new stochastic process, changes as the stock price changes. \bullet n=1 Example 19.4 For x \in E, let $\tau x := \inf\{n > 0 : Xn = x\}$. (ii) If $d \leq 2$, then Px0 [$\tau x1 < \tau x0$] = 12. Corollary 21.41 Let (Xi, i \in I) and (Yi, i \in I) be families of random variables in C([0, ∞)). Define X(ω) := inf{n \in N : $\omega n = 1$ } – 1, where inf $\emptyset = \infty$. In order for the problem to be interesting, assume also that the distribution π cannot be constructed directly too easily., 0) be the first unit vector in Zd. Let I = [a, b] \subset R be an interval and let λ be the Lebesgue measure on I. an 1 + (x/a)2 Then Cauchy distribution with parameter a. Proof Step 1. For J \subset I, let Ω J := $\times \Omega$ j and AJ = Aj . For such z, define $hz \in C$ by hz(y) = f(x) + f(z) - f(x) Hz (y) Hz (z) for all $y \in E$. Indeed, let (Xn) $n \in N$ be an independent family of random variables $n \rightarrow \infty$ with Xn ~ Ber1/n . As in the preceding example, we use a compactness argument. 1.4 Measurable Maps 37 Theorem 1.80 (Composition of maps) Let (Ω , A), (Ω , A) be measurable spaces and let X $: \Omega \to \Omega$ and $X: \Omega \to \Omega$ be measurable maps. Hence $\Xi N = ki=1$ (Ni /N) δei . Partial sums of independent centred random variables are an important example. $K \to \infty$ Then E[YiK] = 0 and Var[ZiK] $\to 0$ as well as Var[YiK] $\leq \sigma 2$, $i \in N$. l=1 In this case, (Xn, l) also satisfies the Lyapunov condition. Dropping the assumption of recurrence is easier, as the following theorem shows. an distribution with Corollary 16.30) If P*X is in the domain of attraction) of a stable * index α , then E $|X|\beta < \infty$ for all $\beta \in (0, \alpha)$ and E $|X|\beta = \infty$ if $\beta > \alpha$ and $\alpha < 2$. 2.1 Independence of Events. (ii) The distribution PX of X is uniquely determined by ψX ., km k1 !··· km ! km is the multinomial coefficient and pk = $p1k1 \cdots pm$. (iii) Cauchy distribution: Caur * Caus = Caur+s for all r, s > 0. Proof (i) Let $x \in Rn$ and $A := \{(u, v) \in Rn \times Rn : u + v \le x\}$. be measurable maps with the property that $fn \rightarrow n \rightarrow \infty f$, but not $fn \rightarrow n \rightarrow \infty f$, but not $fn \rightarrow n \rightarrow \infty f$, but not $fn \rightarrow n \rightarrow \infty f$. Be measurable maps with the property that $fn \rightarrow n \rightarrow \infty f$. ≥ 0 and $g \geq 0$ and (to avoid trivialities) f + gp > 0. 26.4 for a computer simulation of Feller's branching diffusion., 0, 1) are left eigenvectors for the eigenvalue 1. Then $m + kd \in N(x, y)$ for every $k \geq nx$. \blacklozenge Example 7.13. Show that Wt := e - t Xt, $t \geq 0$, is a martingale. + Xn - 1), *) 1 1 E Yn - 1 F - n = X(1) + . In the special case of the Fourier basis b0 (x) = 1 and bn (x) = 2 cos(n\pi x), n \in N, this construction goes back to Paley and Wiener [125,3 Theorem XLV, page 154]. Choose $J \supset J^{\sim} := J \cup \{j\}$. are measurable, then meas fn \rightarrow f $\iff \tilde{n}$ n $\rightarrow \infty$ d(f, $\rightarrow 0.$, Utn } of the covering U := {Ut, t \in I} of I. Hence Theorem 1.81 yields that Y is measurable. Things would be easy if the individual coordinates of the chain were independent one-dimensional random walks. k *) fi (Xi) A = E fi (X1) A . Remark 1.14 (i) lim inf and lim sup Can be rewritten as lim inf An = $\omega \in \Omega$: #{n $\in N$: $\omega \in An$ } = ∞ . (ii) If (A^{*}i) i \in I is an independent family of σ -algebras and if each Xi is Aⁱ - Ai measurable, then (Xi)i \in I is independent. Now | log(x) - (x - 1)| \leq |x - 1|2 for all x \in C with n $\rightarrow\infty$ |x - 1| \leq 12. In this sense, aperiodicity has the flavour of an irreducibility condition which is needed in order that two independent chains started in arbitrary states could meet each other. For r > 0 let Mr (Xn) = E[|Xn | r] be the rth absolute moment. $\bullet = A \times B$ Example 8.32 Let $\mu 1$, $\mu 2 \in R$, $\sigma 1$, $\sigma 2 > 0$ and let Z1, Z2 be independent and N μi , $\sigma 2 - distributed$ (i = 1, 2). 18.4), the distribution will never be exactly the invariant distribution (i = 1, 2). 18.4), the distribution (i = 1, 2). 18.4), the distributed (i = 1, 2). 18.4), the distribution (i = 1, 2). 18.4), the distributio Hence we have $X \sim PPP\mu$. 8.3 Regular Conditional Distribution 207 Clearly, fX (x) > 0 for PX - a.a. $x \in R$ and fX-1 is the density of the absolutely continuous part of the class of test functions so it can be adapted to the individual problem. P \circ (X In the following, assume that (Ω, A, P) is a probability space and $\tau: \Omega \to \Omega$ is a measurable map. Let $K := \nu(\Omega)/\delta$ and $\delta < \varepsilon/(2K)$. Let Ln = x1 + . Definition 1.9 A class of sets $A, B \in A$ the difference set $B \setminus A$ is a finite union of mutually disjoint sets in A, (ii) A is \cap -closed. Analogously, a CFP ϕ is called infinitely. divisible if, for every $n \in N$, there is a CFP ϕ n such that $\phi = \phi$ nn. Then τ n is a $n \rightarrow \infty$ stopping time and τ n $\downarrow \tau$; hence B τ n $- \rightarrow$ B τ almost surely. Similarly, we get (d/dt) qp t (x, y). First, however, we have to consider a more general situation. (Here Ep [#TL] denotes the expected value of #TL, which we define formally in Chap. Therefore, (Ω , A, P, τ r 5 15 78 68 21 30 165 27 5 58 29 58 2 measure $\mu = q \in Q$ δq . We will show (Ω , A, P, τr) is ergodic $\Leftarrow r$ is irrational. Then E[X2] = E[X] = P[X = 1] = p and thus
Var[X] = p(1 - p). Now (17.8) is implied by P[Xn > a] + *) 1 P[Xn = a] = E f \tau, (X $\tau + m$) m $\geq 0 2$) * 1 = E0 $\phi(\tau, X\tau) 1 \{\tau \leq n\} \geq P0 [\tau \leq n]$. Define fn = f (\cdot , xn). For further reading, consult, e.g., [31, 32] or [74]. Then $\emptyset = \Omega \setminus \Omega \in A$, and hence A is a ring. Theorem 2.31 If X and Y are independent Z-valued random variables, then PX+Y = PX * PY . * 18.3 Markov Chain Monte Carlo Method 445 Exercise 18.2.2 Consider the bivariate process (X, Y) that was constructed from X[~] and Y[~] in Example 18.6. Show that (X, Y) is a coupling with transition matrix p. , Yn are independent and Yi ~ Berp for any i = 1, (That is, in Theorem 3.2(iii), the assumption $\psi(z) < \infty$ for some z > 1 cannot be dropped.) 4.3.2 Poisson Approximation Lemma 3.6 Let μ and $(\mu) n \in \mathbb{N}$. The matrix A can be chosen to be lower triangular. This is the smallest σ -algebra with respect to which all Xi are measurable. $n \rightarrow \infty$ Definition 7.2 Let $p \in [1, \infty]$ and f, f1, f2, . Example 16.2 *n (i) δx is infinitely divisible with $\delta x/n = \delta x$ for every $n \in N$. \bullet Takeaways Assume we are given a pointwise convergent sequence of nonnegative functions., kr }) = dx. (I ndependence) If σ (X) and F are independent, then E[X | F] = E[X]. The conditions of the theorem are fulfilled as $Fn \uparrow \Omega$. 11.3 Example: Branching Process . Proof "(ii) \Rightarrow (iii) \Rightarrow (iii) \Rightarrow (iii) \Rightarrow (iii) \Rightarrow (iv)" This is obvious. For #J = 0, the statement (2.8) holds by assumption of this theorem. N - 1, and the corresponding eigenvectors are the x k from above. Hence also, t (x, y) = pt (x, y) - p 0 t q(ps - p s) (x, y) ds. Exercise 21.1.1 Show the corresponding eigenvectors are the x k from above. claim of Remark 21.7. & Exercise 21.1.2 Let $X = (Xt) t \ge 0$ be a real-valued process with continuous paths. Recall that a finite permutation is a bijection : $I \rightarrow I$ that leaves all but finitely many points unchanged. Theorem 16.14 (Lévy-Khinchin formula on $[0, \infty)$) Let $\mu \in M1$ ($[0, \infty)$) and 3 let $u : [0, \infty) \rightarrow [0, \infty)$, $t \rightarrow -\log e^{-t} x \mu(dx)$ be the log-Laplace transform μ . \bullet Exercise 21.2.3 Let B be a Brownian motion and $\sigma > 0$. Then $* n \rightarrow \infty$) n Ssn $\rightarrow L[Bt - Bs] L St - for any t > s \ge 0$. Consider now the situation where X does not necessarily jump to one of its nearest neighbors but where we still have E0 [[X1]] < ∞ and E0 [[X1]] = 0. If A \subset B, then U(A) \supset U(B); hence $\mu * (A) \le \mu * (B)$. n=1 Hence, by the monotone limit theorem, we can interchange the summation and the integral and obtain) * $\infty \infty \infty$ E Yn2 – 2 \leq n 1{x x] dx n=1 \leq 4 0 ∞ P[|X1 | > x] dx = 4 E[|X1 |]. Further, let Xi : Rn \rightarrow R, (x1 , . Here N0 \subset R is a ν -null set with μ (R \ N0) = 0. Note that qn = ψ n (0) = ψ (qn-1) for all n \in N and qn \uparrow q. Then, for $\varepsilon > 0$ and k, N \in N, choose numbers K ε and δ N,k, ε such that ε sup Pi { $\omega : |\omega(0)| > K\varepsilon$ } $\leq 2i \in I$ and $0 \ 1 \ 1$ N sup Pi $\omega : V(\omega, \delta N, k, \varepsilon) > \leq 2 - N - k - 1 \varepsilon$. 18.4 for a computer simulation of the curve $\beta \rightarrow m(\beta)$. Assume that (fn)n \in N does not converge to f in L1. Hence, for p 23 P[#CN $< \infty$] = $\infty *$) P there is a closed circle $\gamma \in \Gamma$ n = 2N $\leq \infty$ n n $\cdot 3(1 - N)$ irpose, it suffices to show that, for every k = 1, . Step 3. Step 3. Exercise 14.4.4 Show that a continuous real convolution semigroup ($vt : t \ge 0$) with $vt ((-\infty, 0)) = 0$ for some t > 0 is nonnegative. 12.2 Backwards Martingales ... The function I is lower semicontinuous real convolution semigroup ($vt : t \ge 0$) with $vt ((-\infty, 0)) = 0$ for some t > 0 is nonnegative. 12.2 Backwards Martingales ... The function I is lower semicontinuous real convolution semigroup ($vt : t \ge 0$) with $vt ((-\infty, 0)) = 0$ for some t > 0 is nonnegative. 12.2 Backwards Martingales ... The function I is lower semicontinuous real convolution semigroup ($vt : t \ge 0$) with $vt ((-\infty, 0)) = 0$ for some t > 0 is nonnegative. 12.2 Backwards Martingales ... The function I is lower semicontinuous real convolution semigroup ($vt : t \ge 0$) with $vt ((-\infty, 0)) = 0$ for some t > 0 is nonnegative. 12.2 Backwards Martingales ... The function I is lower semicontinuous real convolution semigroup ($vt : t \ge 0$) with $vt ((-\infty, 0)) = 0$ for some t > 0 is nonnegative. minimum at I (0) = 0. Show that Jn \leq 4n+1. Hence these are the natural classes of sets to be considered as events in probability theory. (21.9) As in the proof of Lemma 21.3(iii), we infer (with K := C0 2(1- γ)n0) |Xt (ω) – Xs (ω)| \leq K |t – s| γ for all s, t \in D., Xn–1 and Dn but not on the full information inherent in the values D1, . Or consider bond percolation on Z2. As you see, the argument follows a pattern similar to the proof of Carathéodory's theorem. If, in addition, E is locally compact, then Cc (E) \cap Lip1 (E; [0, 1]) is separating for M(E). Then $\phi X \phi Y = \phi X \phi Z$; hence X + Y = X + Z. As examples, we have shown the reflection principle and Lévy's arcsine law. As a composition of measurable maps, $\omega \rightarrow d(f(\omega), g(\omega))$ is measurable.) Let f, f1, f2, . (i) X is a martingale if and only if, for any locally bounded predictable process H, the stochastic integral H \cdot X is a martingale. We define the transition matrix p^2 on Zd by $p(x, d \text{ for } y - x \in \{-1, 0, 1\}$. 0 In fact, it is obvious that X is centered and Gaussian (since it is a limit of the Gaussian (since it processes of partial sums) and has the given covariance function. (i) For $n \in N$, define $\phi_n = \delta_1/n - \delta_2/n$. Convolution semigroups are a special application and yield real valued processes with independent L(P)-random variables are uncorrelated) Let $X, Y \in L1$ (P) be independent. Assume there exists a measurable f with $fn \rightarrow f\mu$ -almost everywhere. (iii) If $E[|XY|] < \infty$ and Y is measurable with respect to F, then E[XY|F] = Y E[X|F] (iv) (v) (vi) (vii) (vii) and E[Y|F] = Y E[X|F] (iv) (v) (vi) (vii) (viii) and E[Y|F] = Y E[X|F] (iv) (v) (vi) (viii) (viii) and E[Y|F] = Y E[X|F] (v) (v) (vi) (viii) (viii) and E[Y|F] = Y E[X|F] (v) (v) (vi) (viii) (functions C0, Sn, Cn, n \in N form an orthogonal system in L2 ([0, 1], λ). Note, however, that those statements that we make explicitly about martingales (such as (ii) in the preceding theorem). Since μ 1 and μ 2 are finite measures, for the function ρ C, ϵ from Lemma 13.10, we have $0 \leq \rho$ C, ϵ from Lemma 13.10, we have $0 \leq \rho$ C, ϵ from Lemma 13.10, we have $0 \leq \rho$ C, ϵ from Lemma 13.10, \omega fr $\leq 1 \in L1$ (µi), i = 1, 2, for all $\varepsilon > 0$. The technique of coupling from the past allows for drawing exactly according to the desired distribution. Furthermore, X- $\infty = E[X0 \ F-\infty]$, n $\rightarrow \infty$ where F- $\infty = \infty$ F-n., Yi-1] = ni=1 E[Yi2] (as in Example
10.2). Let I be an arbitrary index set. In particular, V is no larger than the multiplicity of the eigenvalue with second largest modulus. 17.4 Discrete Markov Chains: Recurrence and Transience . Exercise 6.2.1 Let $H \in L1$ (µ) with H > 0 µ-a.e. (see Lemma 6.23) and let (E, d) be a separable metric space. If A2L occurs, then there are two points x 1, x 2 on the boundary of BL such that for any i = 1, 2, there is an infinite self-intersection free open path πx i starting at x i that avoids x 3-i. We now come to another 0-1 law for independent events and for independent σ algebras. & Exercise 13.2.2 Show that the topology of weak convergence on Mf (E) is coarser than the topology of weak convergence on Mf (E) is coarser than the topology induced on Rn . n) Theorem 6.24 A family $F \subset L1$ (µ) is uniformly integrable if and only if the following two conditions are fulfilled. Hint: First show that) * P |Xn | > n for infinitely many $n = 0 \iff X1 \in L1$ (P). Define Fn := $\tau A0 < \tau x1$. $x \uparrow 1$ (n) In this case, ψX is uniquely determined by the derivatives ψX (1), $n \in N$. We quote the following strengthening of Cramér's Theorem (see [31, Theorem 2.2.3]). & Exercise 15.4.3 Show that, for $\alpha > 2$, the function $\varphi \alpha$ (t) = e - |t| is not a characteristic function. We will use this Exercise later in Example 17.27. Thus we can make the following definition that is the final definition for the integral of measurable functions. k=1 Since $c \ge 0$, we have (Sk + c)2 1Ak \ge (t + c)2 1Ak (ii) If Xt, t \in I, are i.i.d. random variables, then (Xt) t \in I is stationary. (11.3) $n \in N$ Show that C = A + = A - (mod P). (2-2n+1)s+2n+1 -1 Compare also Lemma 21.44. The effective resistance is Reff (0 \leftrightarrow 6) = R(0, 1) + . Define Xnc := (Xn - ϵ) 1F, ϵ Snc := X0c + . 2 5/2 5/6 5 5 1 5 484 19 Markov Chains and Electrical Networks 2 x 5/2 0 1 2 x 1 19/25 19/6 19/10 0 95/12 19 19 Fig. Some of the material of this chapter is taken from [110] and [36]. We close this section with a generalization of Example 12.15 to mean values of functions of k \in N variables., N, are independent, and hence, we have) n * n $\rightarrow \infty$ L (St1 - Stn0, . i=1 Let n \in N with ∞ µ(Bi) 0, there exist countably many A1, A2, . 2) * 1 Since Asn \uparrow A0n for $\varepsilon \downarrow 0$, there is an $\varepsilon > 0$ with p := P A $\varepsilon 0 \ge 4$ N > 0. Successively choose $\omega 2$, $\omega 3$, . , Mn,m) is multinomially distributed with parameters n and p = (p1, . As for A2L,0, we define A3L,0 as the event where there are three distinct points on the boundary of BL that lie in different infinite open clusters if we consider all edges in EL as closed. In order to show that fa is a density and has the same moments as f, it is enough to show that, for all $n \in N0$, $m(n) := 0 \propto x n f(x) \sin(2\pi \log(x)) dx = 0$. \propto Theorem 17.41 $x = -\infty |x| p(0, x) < \infty$ is recurrent ∞A random walk on Z with if and only if x p(0, x) = 0. As K is compact, there are finitely 13.1 A Topology Primer 275 many points x1, . , jn }). \bullet Recurrence, irreducibility and aperiodicity alone are not sufficient for the independent coalescence coupling to be successful. T⁻tK + (k - sn)YkKn . log(P (xk , xk+1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , x1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({x0 })P(x0 , xn-1) log(P (x0 , x1)) k=0 xk ,...,xn-1 = H (\pi) - (n - 1) \pi({)). 9.2 Martingales 219 Remark 9.25 Clearly, for a martingale, the map $t \rightarrow E[Xt]$ is constant, for submartingales it is monotone increasing and for supermartingales it is monotone decreasing., xn) of points in Zd with)xi-1, xi * \in E p for all i = 1, . * Exercise 1.1.4 Let Ω be an uncountably infinite set and A = σ ($\{\omega\}$: $\omega \in \Omega$). By Step 2. By virtue of the differentiation lemma (Theorem 6.28) and using partial integration, we get d 1 $\phi(t) = \sqrt{dt} 2\pi \propto -\infty$ eit x ix $e-x 2/2 dx = -t \phi(t)$., Xn, d) T \in Rd, n \in N, be random vectors. $\check{}$ Hence X is a random walk with transition matrix p. Indeed, for any n \in N, the random variable Yn is σ (X2n , X2n-1)-measurable by Theorem 1.91, and (σ (X2n, X2n-1))n \in N is independent by Theorem 2.26. The books, often well classtested by their author, may have an informal, personal even experimental approach to their subject matter. Further, let q be a sub-probability distribution; that is, qe ≥ 0 for all $e \in E$ and $e \in E$ and $e \in E$ qe ≤ 1 . Typically, however, traders are risk-averse and thus real market prices include a discount due to the inherent risk. Define X0 = 0 and Xn := Y1 + . , αm , $\beta 1$, . Denote by ψ (n) := ψ (n-1) $\circ \psi$ its nth iterate for n \in N. We thus make the following definition. Then B(Ω, τ) = B A, τ . , gn }. Definition 5.25 (Entropy) Let p = (pe) e \in E be a probability distribution on the countable set E. On the other hand, $h d\mu = \lim k \rightarrow \infty$ $h d\mu nk = h d\nu$ for all $h \in C$; hence $\mu = \nu$. Finally, Theorem 14.50 implies the existence of the process X. Define T := T 1 + . Then simple random walk on (E, K) (see Definition 19.11) is recurrent. be i.i.d. real random variables with symmetric distribution L[Y1] = L[-Y1]. Theorem 1.65 (Approximation theorem for measures) Let $A \subset 2\Omega$ be a semiring and let μ be a measure on σ (A) that is σ -finite on A. 1.2 L Now assume $\mu_1 \leq st \,\mu_2$. i=1 Hence (1.1) holds and the proof is complete. We compute the CFP of CPoirv for this ν , $\phi r \nu$ (t) = exp r ∞ ((1 - p)eit) k - (1 - p)eit (k - r = pr 1 - (1 - p)eit . < tn and α_1 , 21.4 Supplement: Feller Processes 533 Remark 21.23 If F is an arbitrary filtration and Ft_* is the completion of Ft_+ , then F_{*} satisfies the usual conditions. Hence, by the martingale convergence theorem (Theorem 11.7), X converges Px -almost surely to a random variable X ∞ with Ex [X ∞] = Ex [X0] = x. Then J -1 PJ \circ (XL) (A) = PJ (A × E) = A PL d(ω 0, . U \in U on M(E). 4 σ σ (21.35) Case 2: t $\geq n-1$. Let K = R or K = C. These φ contributions are quantified in terms of the tilted probability measures $\mu \varepsilon (dx) = \varphi - 1 \varphi(x)/\varepsilon \mu \varepsilon (dx)$, $\varepsilon > 0$, for which we derive an LDP. By the monotone convergence theorem (Lemma 4.6(ii)), $) * n \rightarrow \infty$) * E 1A Yn E[X |F] $\rightarrow E$ 1A Yn E[X Measures Proof of Prohorov's theorem, Part (i), general case There are two main routes for proving Prohorov's theorem in the general situation. Let Cw(i, j) := 0 if |i - j| = 1 and i - 1 k=0 k, Cw(i + 1, i) := -1 k=i if $i \ge 0$, if i < 0. 1 3 2 5 6 9 7 10 12 Fig. Definition 12.4 (i) A map $f : E n \rightarrow E$ is called symmetric if f = f for all $\in S(n)$ Then $A = \{A \subset R : A \text{ is countable}\}\$ is a σ -ring. Assume that $\phi(t) = 1$ for some t = 0. To this end, we assume that there exists an $x \in E$ such that f(x) > 0 and deduce a contradiction. Due to the monotonicity of the conditional expectation (Theorem 8.14(ii)) it is easy to show that the limit does not depend on the choice of the sequence (Xn) and that it fulfills the conditions of Definition 8.11. If the random variables are independent, then terminal events either have probability 0 or 1 (Kolmogorov's 0-1 law). 0 Then PX is called the exponential distribution with parameter θ (in shorthand, expθ). We start with a simple lemma. We denote by Q the set of strictly positive rational numbers. Theorem 23.11 (Cramér) If X1, X2, μ is lower semicontinuous. $n \rightarrow \infty$ (viii) Let $|Xn| \leq Y$ for any $n \in N$ and $Xn \rightarrow X$ almost surely. In the extreme case of a trivial σ -algebra $A =
\{\emptyset, \Omega\}$, hence $A^* = \{\emptyset, \Omega$ finite subset $J \subset I$ the product formula holds: ' (P Aj = P[Aj]., ωn] is an open set, as it is the union of (#E)n - 1 open balls [$\omega 1$, $n \rightarrow \infty$ Similarly, we can show that $E[X\sigma] \leq E[X\sigma]$. Hence I is also P-trivial and therefore (Xn)n $\in N$ is ergodic. On the other hand, there are Lebesgue integrable functions that are not Riemann integrable (such as 1Q). } where u1 < u2 < .550 21 Brownian Motion Theorem 21.43 (Donsker's invariance principle) In the sense of weak convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)), the distributions of S⁻ n convergence on C([0, ∞)) and X. 614 24 The Poisson Point Process Corollary 24.9 The distribution of a random measure X on E with independent increments is uniquely determined by the family (PX(A), A \in Bb (E)). In the definition of upper semicontinuity, we needed the assumption $\mu(An) < \infty$ since otherwise we would not even have \emptyset -continuity for an example as the counting measure μ on (N, 2N). They are the family (PX(A) = A \in Bb (E)). perfect analytic tool for studying sums of independent random variables (on N0) as these sums translate into products of the generating functions. everywhere, 3 (ii) (Triangle inequality) $f d\mu \leq |f| d\mu$, $n \geq m + \# k \leq n - m$: Sl = Sk for all $l \in \{k + 1, ..., lf \mu is a probability measure, then convergence in <math>\mu$ -measure is also called convergence in with C := sup{E[|Xi|] : i \in I } < ∞ , then {PXi : i \in I } is tight. 526 21 Brownian Motion Theorem 21.15 (Blumenthal's 0-1 law, see [18]) Let B be a Brownian motion and let F = (Ft)t $\geq 0 = \sigma$ (B) be the filtration generated by B. 168 7 Lp -Spaces and the Radon-Nikodym Theorem Corollary 7.8 Let I \subset R be an open interval and let ϕ : I \rightarrow R be a map. (21.32) Then, for $\varepsilon > 0$, *) n *) n 1 1 1 n $\rightarrow \infty$ P St - S⁻tn > $\varepsilon \leq \varepsilon - 2$ E (St - S⁻tn) 2 ≤ 2 E[Y12] = 2 $\rightarrow 0$. 11.2 Martingale Convergence Theorems 249 Example 11.15 Let X be a symmetric simple random walk on Z. On the other hand, we have #(A1 $\cap A2 \cap A3$) = 6, thus P[A1 $\cap A2 \cap A3$] = 1 1 1 1 36 = 6 $\cdot 6 \cdot 6 \cdot 6$. In Theorem 3.7, it was shown that the binomial distribution $bn,\lambda/n$ converges to the Poisson distribution Poi λ . Let Htn := n t ΛT (t -1/n) v0 Hs ds for t ≥ 0 , $n \in N$. (5.17) l=1 Theorem 7.33 (Lebesgue's decomposition theorem) Let μ and ν be σ -finite measures on (Ω, A) . Remark 1.62 Later we will see in Theorem 14.14 that the statement holds even for arbitrary σ -finite measures $\mu 1$, For each $n \in N$, there is a Lipschitz constant 1/ ϵ and $\rho A, \epsilon$ (x) = 1, if $x \in A$, 0, if $d(x, A) \ge \epsilon$. t 2 + Var[Sn (5.11)) * P max{|Sk|: k = 1, . R (1, x) (19.13) R (0, x) + If we knew the effective resistances Reff (0 \leftrightarrow x), Reff (1 \leftrightarrow x) and Re independent. 330 15 Characteristic Functions and the Central Limit Theorem Reflection In the Stone-Weierstraß theorem for the case K = C, it is assumed that the algebra is closed under complex conjugation. Denote by Uf = $x \in \Omega 1$: f is discontinuous, we have lim sup inf Iµ (A \cap En) ≤ $\Omega 1$: f is discontinuous, we have lim sup inf Iµ (A \cap En) ≤ $\Omega 1$: f is discontinuous at x the set of points of lim Iµ (vn) = Iµ (v). (Compare Lemma 3.6 and the comment below it.) & Chapter 4 The Integral Based on the notions of measures paces and measurable maps, we introduce the integral of a measurable maps, we introduce the integral Based on the notions of measure spaces and measures are spaces and measures are spaces and measurable maps. integrable. Show that ψ is the Laplace transform of a probability measure μ on $[0, \infty)$ and $n \rightarrow \infty$ that $\mu n \rightarrow \mu$ weakly. = Xn = 0] = n and thus the claim follows. The next two sections are devoted to two applications: The Poisson approximation theorem and a simple investigation of Galton-Watson branching processes. We say that a filtration F satisfies the usual conditions (from the French conditions habituelles) if F is right continuous and if F0 is P-complete. Proof Obvious, since Y -1 (A) \subset A. n=1 (ii) There is a measurable f with ∞ $\mu(A \cap \{d(f, fn) > \epsilon\}) < \infty$ for all $\epsilon > 0$ n=1 and for all $A \in A$ with $\mu(A) < \infty$. Assume that our computer has a random number generator that provides realizations of i.i.d. random variables U1, U2, . Hence p is irreducible. We want to shed some more light on the connection between weak and vague convergence. (iii) W is a closed linear subspace of L2 ([0, 1], λ). Let F \subset A be a σ -algebra. i,j = 1 i=j)m i=1 Xi * = m i=1 Var[Xi]. What is λ 2 in the periodic case? For example, let A be σ $-\cap$ -closed and let (reminder: (Ai)c = A1, A2, . * Exercise 13.1.2 Let μ be a locally finite measure. Then $x \to f(|x|)$ is convex; hence, by Jensen's inequality, **)) E f (|Xi,j| = E f E[Xi |Fj] $\leq L < \infty$. Dr John Preater did a great job language editing the English manuscript and also pointing out numerous mathematical flaws. Then $\mu(A) = \infty$ $\mu(Ai \setminus Ai - 1)$ $= \lim n \rightarrow \infty i = 1$ n $\mu(Ai \setminus Ai - 1) = \lim \mu(An)$. $\phi aX + b$ (t) $= \phi X$ (at) ei)b,t * for all $a \in R$ and $b \in Rd$. (17.11) For example, choose (Rn (x), $x \in E$, $n \in N$) as an independent family of random variables with values in E and distributions P[Rn (x) = y] = p(x, y) for all $x, y \in E$ and $n \in N0$. 12.3 De Finetti's Theorem 271 Exercise 12.3.1 Let (Xn) $n \in Z$ be an exchangeable family of $\{0, 1\}$ -valued random variables. Finally, we consider infinite products of probability spaces. Proof If Y is measurable, then by Theorem 1.80 every Xi \circ Y is measurable. Let (Tti)t ≥ 0 , i = 1, . Evidently, Pz [A] = ∞ z = z p(z, z + 1) $\geq 1 - \infty$ z = z 1 1-z. The following theorem 1.61. For any n \in N, In is uniformly distributed on Λ and Nn is uniformly distributed on the set N := {i \in Zd : i2 = 1} of the 2d nearest neighbors of the origin. Finally, using Kolmogorov's moment criterion for tightness, we show convergence in the path space C([0, ∞)). We thus need a notion of conditional probabilities that allows us to deal with conditioning on events with probability zero and that is consistent with our intuition. Clearly, C p (x) \subset C
p (x) for any x \in Zd ; hence θ (p) $\leq \theta$ (p). Then (fF : F \in I) is uniformly integrable (with respect C C to μ). 1.3 The Measure Extension Theorem 21 (iii) Let A1 , A2 , . Now let $\mu = \delta 0$; hence u(1) > 0. 7 Lp -Spaces and the Radon-Nikodym Theorem 184 Corollary 7.45 Let $\phi \in I$ $M \pm (\Omega, A)$ and let $\phi = \phi + - \phi - be$ the Jordan decomposition of ϕ . Let X[^] and X^{*} be independent random walks with transition matrices p[^] and p, respectively. m \in Z Reflection Check that PX+Y = PX *PY does not imply that X and Y be independent. We interpret Hn as the number of euros we bet in the nth game. Choose a $k \in N$ with $k > 2\gamma 2 - 1$. If the first proposal is rejected, the game starts afresh with proposal X2 and so on. Note that, for the Gibbs sampler also it is enough to know the values of the distribution π only up to the normalising constant. For $\epsilon > 0$ and $k \in \mathbb{N}$, define $g \epsilon := g \land g(\epsilon), f \epsilon := f 1 \{ f \ge \epsilon \}$ and fixe $r \ge -k \otimes \mu(\{ f \epsilon \ge n2-k \})$. However, note that it is Stni – Stni – 1, i = 1, (iii) A is \cup -closed. $\epsilon \downarrow 0 \& 21.5$ Construction via L2 -Approximation 543 Exercise 21.5.2 (Compare Example 8.32.) Fix T \in (0, 1). p 2.4 Example: Percolation 77 Lower bound First we show pc $\geq 1 2d-1$. For example, we are interested in a two-stage random experiment. Further, let C2 \in C with C2 \subset A \cap C1c and α (C2) > β (A \cap C1c) - ϵ . The question that we answer at the end of this chapter is: Is a random walk on the infinite open cluster recurrent or transient? \bullet Takeaways Laws of large numbers show that sums of very many random variables approach their expected value. t = 0 We use an induction argument to show (17.33). In particular, we have P[Mni, T, l > 0] = 1 · $e-\lambda/ni = pi$ and thus Nni ,T ~ bni ,pi , i = 1, 2. Mn, $i := \# l \le n : ti-1 < Xl \le ti = l=1$ By Exercise 2.2.3, the vector (Mn,1, . The implication (iii) \Rightarrow (iv) is straightforward as the characteristic function ϕ uniquely determines the distribution of X by Theorem 15.9. Remark 15.56 For one-dimensional normal distributions, it is natural to define the degenerate normal distribution by Nµ,0 := $\delta µ$., d - 1} and x ∈ Ei. and sup{Var[Xi]: i ∈ I} < ∞, 156 6 Convergence Theorems Proof Since E[Xi2] = E[Xi]2 + Var[Xi], i ∈ I, is bounded, this follows from Corollary 6.21 with p = 2. Clearly, we have P = u(x) = R (0, x). If the limiting function is continuous at 0, then by Levy's theorem, tightness and hence weak convergence are automatic. Indeed, in each case, there is a mass defect in the limit (in the case of the Fn on the left and in the case of the Fn on the left and in the case of the Fn on the left and in the case of the Sn on the right). Note that also $p' := (p - \lambda p)/(1 - \lambda)$ is the transition matrix of a random walk on Zd and that $p = \lambda p^2 + (1 - \lambda)p$. By the scaling property of Brownian motion, $P[A] = \inf P[As] \ge P[B1 - \lambda p)/(1 - \lambda)$ \geq K] > 0 s>0 and thus P[A] = 1. (v) By the addition theorem for trigonometric functions, $1 - \cos(t, X^*) \geq 4 - \cos(t, X^*) = 2 - \cos(t, X^*) = 2$ (Composition of kernels) Let (Ωi , Ai) be measurable spaces, i = 0, 1, 2, and let κi be a substochastic kernel from ($\Omega i - 1$, Ai - 1) to (Ωi , Ai), i = 1, 2. Such a process is called an equivalent martingale, and PX is called an equivalent martingale, and PX is called an equivalent martingale measure. $\pi(x)$ q(x, y) Otherwise the chain X stays at x. For every $n \in N$, choose a covering $Fn \in U(An)$ such that $\mu(F) \leq \mu * (An) + \epsilon 2 - n$. \bullet Theorem 4.15 We have $g \in L1$ (μ) if and only if (gf) $\in L1$ (μ). Further, let (Xi : i $\in I$) be a family of measurable maps Xi : $\Omega \rightarrow \Omega$ i with $A = \sigma$ (Xi : i $\in I$). A real random variable X is called infinitely divisible if, for every $n \in N$, there exist i.i.d. D random variables Xn, 1, . a Show that almost surely $\tau_a, b < \infty$ and that $P[B\tau a, b = b] = -b - a \cdot pn - 1 (x, z)p(z, y) 400 17$ Markov Chains By induction, we get the Chapman-Kolmogorov equation (see (14.15)) for all m, $n \in N0$ and x, $y \in E$, p(m+n)(x, z) = p(m)(x, z) p(n)(z, y). #BL #BL L Now r = (#BL) - 1 Ep $[\#TL] \le d/L$ implies r = 0. Definition 9.44 Let $T \in N$, $a \in (-1, 0)$ and b > 0 as well as $p \in (0, 1)$. Hence the model as $p \in (0, 1)$. is also p = p * := a - b arbitrage-free (for all $p \in (0, 1)$). Evidently, Theorem 20.19 We have lim $n \rightarrow \infty$ Rn = # k $\leq n : Sl = Sk$ for all $l \in \{k + 1, . Let Xn := ni = 1 Yi$ for $n \in N0$. For example, we could not encode one symbol with 011011. \blacklozenge 222 9 Martingales Takeaways A martingale is a mathematical model for a fair game of many rounds. $\sigma \sigma 2 \leq (21.37)$ By (21.35) and (21.37), for every N > 0, there exists a C = C(N, $\sigma 2$) such that, for every n \in N and all s, t $\in [0, N]$, we have *) n, n $\stackrel{-}{\times}$ Kn , n) $4 \leq C t 3/2$. Hence, by Theorem 8.37, there exists a regular conditional distribution of (Xn+k) n $\in \mathbb{N}$ be the infinite product space and let A be the σ -algebra generated by the cylinder sets (see \otimes N (1.8)). * 12.2 Backwards Martingales The concepts of filtration and martingale do not require the index set I (interpreted as time) to be a subset of [0, ∞). * Exercise 13.2.11 For each $n \in N$, let Xn be a geometrically distributed random variable with parameter pn \in (0, 1). is exchangeable, for any choice of filtration and martingale do not require the index set I (interpreted as time) to be a subset of [0, ∞). pairwise distinct numbers $1 \le i1$, Hence we can consider the case I = -N0. By monotonicity of gx, we have $D - \phi(x) \le D + \phi(x)$. Now, for K > 0, compute (using Markov's inequality and Fubini's theorem) Pn $[-K, K]c \le \alpha - 1 = \alpha - 1 =$ cos(tx/K) Pn (dx) dt 1 - Re(ϕ n (t/K)) dt. By the monotone convergence theorem (14.6) and (14.7) also hold for this f. Note that this definition is independent of the representatives of f and g. Therefore, it is enough to prove the existence of an X such that (21.25) holds. Hence, for m < n, we have E n \ E n - 1 also hold for this f. C C For $m \le n$, let Em m n n m and thus $n - 1 \ge m$ is minimum; hence $\phi(\tau) = 0$. Now this causes trouble in many places and so we chose a definition where this implication holds. (iii) Assume that A1, A2, Consider the events $A = \{sup\{Xt : t \in I\} > K - \infty \}$ 5} and $B = \{\sup\{Xt : t \in I\} > K + 5\}$. There are functions $hi : \Omega i \to (0, \infty)$ such that $hi d\mu i < \infty$, i = 1, 2 (see Lemma 6.23). As X is irreducible, we have $\pi(\{x\}) > 0$ for all $x \in E$. Deduce the optional stopping theorem for right continuous supermartingales: $(X\tau \wedge t)t \ge 0$ is a right continuous supermartingale. Theorem 21.6 (Kolmogorov-Chentsov) Let X = 1, 2 (see Lemma 6.23). As X is irreducible, we have $\pi(\{x\}) > 0$ for all $x \in E$. $(Xt, t \in [0, \infty))$ be a realvalued process. 19.2 Reversible Markov Chains 465 Since F (x, y) > 0 for all x, y \in E, we have n0 := min n \in N0 : pn (x0, y) > 0 for all x, y \in E, we have n0 := min n \in N0 : pn (x0, y) > 0 for all x, y \in E, we have n0 := min n \in N0 (for finite $J \subset I$) fJ(x) = -1 ($xj - \mu j$) $2\pi\sigma j 2 \exp - 2\sigma j 2 j \in J j \in J$ for $x \in RJ$. The concept of conditional probabilities and conditional probabilities and conditional expectations formalizes the corresponding calculus. Here, the argument is similar to the proof of Theorem 7.9. Let $g \in L(\phi)$ with $g \in [X1]$, Here we do not want to stress this point but state that, vaguely speaking, all sets that can be constructed explicitly are Borel sets. Let $B \in E$ and $\varepsilon > 0.22222$ In order to obtain the minima of F β , we compute the derivative $! 0 = d\beta dm F(m) = -m - h + \beta - 1$ arc tanh(m). 24.1 Poisson point process on the unit square with intensity measure 50 λ . Clearly, $[\omega 1, (1.10) We write M(\mu *) = \{A \in 2\Omega : A \text{ is } \mu * -measurable\}$. (16.6) In this case, the pair (α , ν) is unique. \clubsuit Chapter 2 Independence The measure theory from the preceding chapter is a linear theory that could not describe the dependence structure of events or random variables. Consider the following five properties. Proof We have to show that, for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $t \in \mathbb{R}d$, all satisfies the dependence structure of events or random variables. \in Rd with $|t - s| < \delta$ and all $\mu \in F$, we have $|\phi\mu(t) - \phi\mu(s)| < \epsilon$. Show that $1 \ n \to \infty f \circ T \ k \to 0$ n n-1 in Lp (λ). Assume that, given Y, the random variables (Xi)i \in I are independent and BerY -distributed. Define the (ultra-)metric d on Ω by $d(\omega, \omega) = 2 - \inf\{n \in N: \omega n = \omega, 0, \text{ if } \omega = \omega, 0, \text{ if } \omega = \omega , 0$ there is an f with fnk \rightarrow f almost everywhere; hence, in k $\rightarrow\infty$ particular, $\mu(\{d(fnk, f) > \epsilon/2\}) \rightarrow 0$ for all $\epsilon > 0$., BtN) = EB_T [F (B)]. "(iii) \Rightarrow (i)" The supremum of convex functions is convex. The converse is false in general., Cnmn \in A such that En \ A = En \ Bn = Cn. Now let K \subset E be compact. Reflection Come up with an example of a measurable function f such that the
integrals on the right hand side of (14.7) both exist but do not coincide. Thus we can find finitely many x1, Example 1.39 (Lebesgue measure) Let $n \in \mathbb{N}$ and let $A = \{(a, b] : a, b \in \mathbb{R} \ a \leq b\}$ 1.3 The Measure Extension Theorem 19 be the semiring of half open rectangles (a, b] $\subset \mathbb{R}$ (see (1.5)). (13.14) Let $\varepsilon > 0$. Chapter 24 The Poisson Point Processes can be used as a cornerstone in the construction of very different stochastic objects such as, for example, infinitely divisible distributions, Markov processes with complex dynamics, objects of stochastic objects such as a cornerstone in the average energy of a particle is 1 U (m) = -m2 - hm. Then D is a π -system \Rightarrow D is a σ -algebra. x,y \in E Therefore (since D \equiv 0), LJ = LI + 1 D(x, y) 2 R(x, y) > LI. Definition 19.17 We call Ceff (A0 \leftrightarrow A1) := I (A1) the effective conductance 1 between A0 and A1 and Reff (A0 \leftrightarrow A1) := I (A the effective resistance between 1) A0 and A1. In the second step, we distribute these jumps uniformly and independently on (0, 1]. Theorem 17.9 A stochastic process $X = (Xt) t \in I$ is a Markov process if and only if there exists a stochastic kernel $\kappa : E \times B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that, for every bounded $B(E) \otimes I \rightarrow [0, 1]$ such that $[0, 1] \otimes I \rightarrow$ Kolmogorov's extension theorem. & Exercise 16.2.2 Show that the distribution on R with density f(x) = is not infinitely divisible. A random vector X = (X1, .) Then CPoiµn $\rightarrow \mu$. (a) We have (f + g) - f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -(f + g) - = f + g = f + -f - +g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g + -g - ; hence (f + g) - +f + g - g - ; hence (f + g) - +f + g - g - ; hence (f + g) - +f - g - g - ; hence (f + g) - g - ; hence (f + g) - g - g - ; hence (f + g) - g - ; hence (fa null set, if $\mu(A) = 0$., LYn and use Remark 15.7 to compute LY1 +...+Yn. 90 3 Generating Functions In particular, for any $n \ge n0$, we have $\mu n (\{N + 1, N + 2, ..., 95, 4.2, Monotone Convergence and Fatou's Lemma ... On the other hand, the Lebesgue integral respects the geometry of the range by being defined via slimmer and slimmer horizontal$ strips. However, this condition is not sufficient for the existence of weak limit points, as for example the sequence (δn) $n \in N$ of probability measures on R does not have a weak limit point (although it converges vaguely to the zero measure). Define $M := n \in N$ $\Omega n -$. By the fundamental theorem of calculus (see Exercise 13.1.7), we have $n \rightarrow \infty$ Htn (ω) $-\rightarrow$ Ht (ω) for λ -almost all t \in [0, T] and for all $\omega \in \Omega$. We have seen that in this case, we must have an = n1/ α for some $\alpha \in (0, 2]$. Similar inversion formulas hold for measures μ on Rd. This implies the second inequality in (23.15). and U1, U2, As an alternative to the backwards martingale argument of Sect. Then we consider backwards martingales and prove the convergence theorem for them. Let X be a random variable that is uniformly distributed on [0, 1]. 3 Remark 14.24 In the following, we often write $\kappa(\omega 1, \omega 2)$ instead of 3 f ($\omega 1, \omega 2$) is not allows us to write the integrator closer to the corresponding in Clearly, F (u) = 1 and for any $x \in V$, we have F (x - F(x)u) = F (x) – F (x) $u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem) $x - F(x)u \in W$ and thus 174 7 Lp -Spaces and the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon-Nikodym Theorem)x - F(x)u = 0; the function of the Radon Nikodym Theorem)x - F(x)u = 0; the function of the Radon Nikodym Theorem)x - F(x)u = 0; the function of the Radon Nikodym Theorem)x - F(x)u = 0; the function of the Radon Nikodym Theorem)x - F(x)u = 0; the function of the Radon Nikodym Theorem)x - F(x)u = 0 $E[Zn] = 0. \in F$ such that $F = \sigma$ ({A1, A2, . Furthermore, $\mu * \nu = \nu * \mu$ and ($\mu * \nu$)(Rn) = $\mu(Rn) \nu(Rn)$. Let r, s > 0 and let Z ~ Γ 1,r+s and B ~ β r,s be independent (see Example 1.107). Denote the shift by τ ; that is, Xn = X0 ° τ n. We will decompose X into a sum consisting of a martingale and a predictable process. Klenke, Probability Theory, Universitext, 303 304 14 Probability Measures on Product Spaces 14.1 Product Spaces 14.1 Product space)
Let (Ω , i \in I) bean arbitrary family of sets. This procedure leads to a generalisation of the concept of a product measure. By Remark 1.17(ii), we conclude that $\delta(E) \subset DE$ for any $E \in E$. \bigstar 17.2 Discrete Markov Chains: Examples 399 Takeaways For Markov processes, the future depends upon the information up to a given time only via the state at this very time. Since μ is additive, we have $\mu(Ck, i) + i=1 \text{ dk } \mu(Ck, i) > i=1 \text{ ck }$ showing that there exists an open set $U \supset K$ and a $\delta > 0$ such that $\rho K, \epsilon \leq 1U \in L1$ (µi), i = 1, 2 for all $\epsilon \in (0, \delta)$. TV (18.7) (iii) Equation (18.7) holds for some $x \in E$. Hence, it is enough to consider (Xn +)n $\in N$. Since VT is FT -measurable, by the factorization lemma (Corollary 1.97) there exists a function $qT : RT \rightarrow R$ with VT = qT (X1, . 21.5) Construction via L2 -Approximation .. By (9.18), $\tau \wedge t$ and $\sigma \wedge t$ are stopping times for any $t \in I$. Two functions f, $g \in H$ are considered equal if $f = g \lambda$ -a.e. Let (bn) $n \in N$ be an orthonormal basis (ONB) of H ; that is,)bm , bn * = 1{m=n} and n ; ; ; ;)f, bm *bm ; = 0 lim ; $f - n \rightarrow \infty$ for all $f \in H$. 425 In this case, p n (x, y)p(y, z) = $y \in E'$ (Px Xn = y, $\tau x 1 > n$, $Xn+1 = z \neq E'$ (= $Px \forall x_1 > n + 1$; Xn+1 = z = p n+1 (x, z). *n If μ is infinitely divisible and $\mu \in M1$ (R) is such that $\mu = \mu$ for all $n \in N$, then $\nu n = 1R \setminus \{0\}$ num is a possible choice. This chapter is devoted to a systematic treatment of almost sure convergence, convergence in measure and convergence of integrals. , n} with s, t \in Uti. \bullet Takeaways A measure preserving dynamical system consists of a probability space (Ω , A, P) and a measure preserving map τ : $\Omega \rightarrow \Omega$. This is the starting point for the investigation of topological properties of Lp spaces in the subsequent sections. Thus PX is uniquely determined by its 3 moments E[Xn] = x n PX (dx), n \in N. If in addition E is locally compact (e.g., E = Rd), then one can even show that $n \rightarrow \infty$ $\Xi n \rightarrow \Xi \infty$ almost surely. Therefore, AI $\subset \sigma$ (Z E , R). In contrast with σ -algebras, topologies are closed under finite intersections only, but they are also closed under finite intersections only. Let X1 , X2 , . , n (since mn+1 = 0), $\phi\mu$ (al) = n k=1 n al + ak (mk+1 - mk) 1 - = (ak - al)(mk+1 - mk) ak k=l n' (= (an - al)mn+1 - (al - al)ml - (ak - ak-1)mk k=l+1 = - n (yk - yk-1) = yl = \phi(al). Corollary 15.3 Let E be a compact metric space. Hence the expected length is - pe log2 (pe) \leq Lp (C) \leq 1 - pe log2 (pe). Proof The second part of the theorem was shown in the above construction. Similarly, we define $\delta(E)$ as the λ -system generated by E. For sufficiently large |n|, the sets A ϵ and $\tau - n$ (B ϵ) dependent. " \subset " Clearly, every Xt : $\Omega \rightarrow R$ is continuous and hence (B($\Omega, d) - B(R)$)measurable. 23.4 Varadhan's Lemma and Free Energy. If p = I then any function f that coincides with q on A is a solution of the Dirichlet problem. The analogous statement holds for any of the classes E1, . (iii) For any $r \in (0, 1)$, ψX is uniquely determined by countably many values $\psi X(x_i)$, $x_i \in [0, r]$, $i \in N$. x_1 , x_2 , $x_3 \in BL \setminus BL-1$ mutually distinct Let L be large enough for Pp $[A3L, 0] \ge Pp [N \ge 3]/2 > 0$. We call $\kappa 1 \otimes \kappa^2$ the product of κ_1 and κ_2 . Show that (Bt2 -t)t ≥ 0 is a martingale. Then (14.6) and (14.7) hold trivially. This piece is necessarily of dimension smaller than n. Hence, for fixed t, we have lim ft (x) = - = ft (0). Then the independent coalescent chain is a successful coupling. We consider the problem of sampling a random variable Y with distribution π on a computer. Hence a probability measure μ is uniquely determined by the value $\mu({1})$. We write X = Y if PX = PY (D for distribution)., x) = i=1 The distribution function of the random variable Z := min(X1, . Uniformly integrable (sub-, super-) martingales converge almost surely and in L1 . 136 5 Moments and Laws of Large Numbers Proof We decompose the probability space according to the first time τ at which the partial sums exceed the value t. (In Theorem 8.20, we will see that we have $E[Y 2] \leq E[X 2]$, but here we want to keep the proof selfcontained.) Let Y be F -measurable and assume $E[Y 2] \leq \infty$. Clearly, τ (n) is the identity map, hence hn = h2n = . (ii) For $\delta > 0$, let $Y\delta \sim U[1-\delta,1]$ be independent of X. or its affiliates An online phone book, provides a quick way to look up numbers of people and businesses you want to call or locate. Clearly, fy, $\uparrow 1\{0 \rightarrow p y\}$ for $n \rightarrow \infty$; hence it suffices to show that each fy, n is measurable. Formally, we sometimes write $Var[X] = \infty$ if $E[X2] = \infty$. (vi) If $Xn \ge 0$ almost surely for all $n \in N$, then E n=1 n=1 (vii) If $Zn \uparrow Z$ for some Z, then $E[Z] = \lim n \to \infty E[Zn] \in (-\infty, \infty]$. Then there exists a probability space (Ω, A, P) with random variables X, X1, X2, Lemma 15.20 Let $\mu \in M1$ (Rd) with characteristic function ϕ . Here the state space is $E = \{0, ... \bullet Theorem 15.57 (Cramér-Wold Provide Provide$ device) Let Xn = (Xn, 1), Quite the contrary is suggested by the many invitations to play all kinds of "sure winning systems" in lotteries. Hence, now let X be a martingale with |Xn - Xn - 1| = 1 almost surely for all $n \in N$ and with $X0 = x0 \in Z$ almost surely. Since $x \to d(x, z)$ is continuous and hence measurable, the maps fz, $gz : \Omega \to [0, \infty)$ with \mathbb{C} The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2020 A., Xk) = ∞ (Y1, Analogously, we can define the conditional expectation X + \in L1 (P). Clearly, μ 1 \leq st μ 2 implies F1 (x) \geq F2 (x) for all $x \in \mathbb{R}^d$. (ii) If $\Omega = \mathbb{R}$ and $A = B(\mathbb{R})$ is the Borel σ -algebra on \mathbb{R} , then X: (Ω, A) $\rightarrow \mathbb{R}$, B(\mathbb{R}) is called an Ameasurable real map. Theorem 19.12 If X is a reversible Markov chain and if π is an invariant measure, then X is a random walk on E with weights $C(x, y) = p(x, y) \pi(\{x\})$. Without loss of generality, we can assume that E[X1] = 0. Let $f = d\nu d\mu$ be the Radon-Nikodym derivative and let $I = \{F \subset A : F \text{ is a } \sigma \text{ -algebra}\}$. Construct a counterexample that shows that right continuity of the paths of X is essential. We prepare for the proof of this theorem with a couple of lemmas. Before we introduce characteristic functions (and Laplace transforms) later in the book, we want to illustrate the basic idea with probability generating functions that are designed for N0 -valued random variables. (5.12) Furthermore, Kolmogorov's inequality holds: a generalization of Kolmogorov's inequality. By monotone convergence, we get $E[X] = -\lim F(\lambda) \lambda \downarrow 0$ for all $n \in N$. We conclude $q = \min F$. On the other hand, we have 1 - q > 0 on A; hence $\mu(A) + \nu(A) = 0$ and thus $\nu_a(A) = \nu(A) = 0$. With our machinery, so far we can define the conditional probability P[A|X] for fixed $A \in A$ only. Recall the definition of a transition kernel from Definition 8.25. Let $K \in R$ and let $\tau = \inf\{t : Xt \ge K\}$ be the stopping time of first entrance in $[K, \infty)$. The full proof of the general case is deferred to the end of the section. , Xn are independent and N0 -valued random variables, then $\psi X1 + ... + Xn = n \ \psi Xi + E[Xn \ Fm] = X1 + .. (n + 1)N - 1\}$, define $Yk := Y \ Y \ n \ n \ n + 1$, $Uk - nN \ Xk$, if $n < \tau$, else. The number of clicks should obey the following rules. Definition 17.54 Let (E1, E1, μ 1) and (E2, E2, μ 2) be probability spaces. This information can be measured by the length of the shortest sequence of zeros and ones by which the message can be encoded. 282 13 Convergence of Measures Remark 13.13 By Theorem 13.11, the weak limit is unique. Finite additivity follows from additivity of the integral (Lemma 4.6(iii)). Hence (and since each of the families $\{X + , Y + \}$, $\{X - , Y + \}$, $\{X - , Y + \}$, $\{X - , Y - \}$ is independent) we obtain E[XY] = E[(X + - X -)(Y + - Y -)] = E[(X + - X -)(Y + - Y -)]E[X + Y +] - E[X - Y +] - E[X + Y -] + E[X - Y -] = E[X +] E[Y +] - E[X -] E[Y +] - E[X -] E[Y -] = E[X +]1, Lemma 7.15 (Young's inequality) For p, $q \in (1, \infty)$ with x, $y \in [0, \infty)$, $xy \le 1 p + 1 q = 1$ and for yq xp + 1. Exercise 17.3.1 Consider the Yule process X from Example 17.27. A B $n \rightarrow \infty$ Choose a sequence (gn) $n \in N$ in G such that gn $d\mu \rightarrow \gamma$, and define the function fn = g1 v. Takeaways The distributions of the arithmetic means of a growing number of i.i.d. random variables concentrate more and more around the expected value (under certain regularity assumptions, that is). (ii) If $C \subset Rn$ is a closed set, then $C \in \tau$ is in B(Rn) and hence C is a Borel set. 9-15. 2.4 Example: Percolation . Now arc tanh(z) = 1+z 1 log for $z \in (-1, 1)$ and 21-z 1 1 cosh arc tanh(z) = $\sqrt{-1}$. Definition 1.59 (Distribution function) A right continuous monotone increasing function $F: R \to [0, 1]$ with $F(-\infty) := \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty)
:= x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$ and $F(\infty) := x \to -\infty \lim F(x) = 0$. A if the generated σ -algebras (σ (Xi))i \in I are independent given A (and the conditional distributions P[Xi $\in \cdot |A]$ are equal). Proof The implications (iv) \Rightarrow (ii) \Rightarrow (iii) are evident. Then the difference chain (Zn)n \in NO is a symmetric random walk 18.2 Coupling and Convergence Theorem 441 with finite expectation and hence recurrent. Then clearly D(A0) = D(A1) = 0. The effective resistance is Reff $(0 \leftrightarrow 1) = (R1-1 + . \blacklozenge 0.05 \cdot 0.02 + 0.9 \cdot 0.98 \ 883$ Now let $X \in L1$ (P). Definition 17.29 For any $x \in E$, let $\tau x := \tau x1 := \inf\{n > \tau xk = 1 \ xn = x \ for \ k \in N$, $k \ge 2$. We show that, along the sequence $(xn) n \in N$, the difference $n \rightarrow \infty$ quotients converge n=0 Its distribution function is F (x) = 1 - (1 - p) x+1! v0 for x \in R. If in addition μ and μ are measures on (E, E) and (E) are measures on (E) are $\Lambda \uparrow A1$. The Stone-Weierstraß theorem and its corollaries allow to boil down the class of test functions to a tractable size. Definition 8.2 (Conditional probability) Let (Ω , A, P) be a probability space and $B \in A$. $\infty \propto Ak$. In the next section, we will see that in the current example this can be done using transition kernels. Then, for $t \in (0, \theta)$,) * E et X $= \theta \propto 0$ et x $e - \theta x$ dx $= \theta < \infty$. (i) Show that almost surely 3 E[h(X)|Y] = h(x)f(x, Y) \lambda(dx) 3 . n=0 Inductively, the claim follows for all $n \ge 2$. (i) Show that the set $M := \{\mu \in M1 \ (\Omega) : \mu \circ \tau - 1 = \mu\}$ of τ -invariant measures is convex. Hence $F \le \delta/\epsilon < \infty$. Var[Sn]1+($\delta/2$) l=1 Example 15.43 Let (Yn)n \in N be i.i.d. with E[Yn] = 0 and Var[Yn] = 1. As in Example 20.33 we obtain hn (P, τr ; P) = hkn (P, τr ; P) = hkn (P, τr ; P) = 0, n - 1, the distribution on (Ωi , A i) depends on ($\omega 1$, . The chain thus has period 2. (iii) Assume A1, A2, . i=1 n $\rightarrow \infty$ By assumption, limn $\rightarrow \infty$ Fn (∞) = F (∞) and Fn (γi) $-\rightarrow$ F (γi) for every i = 0, . Furthermore, M($\mu *$) = σ (A \cup N μ) = {A \cup N : A \in A, N \in N μ } and $\mu * (A \cup N) = \mu(A)$ for any $A \in A$ and $N \in N\mu$. As shown in the first step, for any $n \in N$, there is an open set $Wn \supset Cn - n - 1$. Hence, there exists a relatively compact $Ux \in U$ with $x \in Ux$. (See [54, Chapter XVII.2a, page 565].) \blacklozenge Takeaways A random variable with finite nth moment possesses a characteristic function that is n-times differentiable. At each step, a ball is drawn and is returned to the urn together with an additional ball of the same color. As shown in Theorem 17.50, for this uniqueness it is sufficient that the chain be irreducible. This method is called coupling from the past and goes back to Propp and Wilson [138] (see also [55, 56, 92, 137, 139, 171]). , $\omega n-1$, ωn = PL (A). The map $\Lambda *$ is a convex rate function but is, in general, not a good rate function. Theorem 13.33 (Helly's theorem) Let (Fn)n \in N be a uniformly bounded sequence in V. Indeed, for almost all $x \in [0, 1]$, P[Y $\in \cdot |X = x] = (Berx) \otimes n$. Theorem 1.16 (Generated σ -algebra) Let $E \subset 2\Omega$. Let $\omega 0$ = (0, 0, . For a nice exposition including many examples, see also [99]. $2 2\pi - \infty 23.1$ Cramér's Theorem 593 Furthermore, $t_2 z_2 . 348 15$ Characteristic Functions and the Central Limit Theorem Froof We have n yky $\frac{1}{4}(t_k - t_1) = k_l = 1$ n yk eixtk yl eixtl $\mu(dx) k_l = 1 = n$ yk eixtk $\mu(dx) \ge 0$. (ii) Assume there is a metric that induces weak convergence. .}) + μ n ({k}) - μ ({k}) k \in A main goal of this section is to show that every infinitely divisible distributions N μ , σ 2 with $\mu \in \mathbb{R}$ and σ 2 > 0, and • (limits of) convolutions of Poisson distributions. We assume that Pp $[N \ge 3] > 0$ and show that this leads to a contradiction. Assume that there is a sequence (tn)n \in N of real numbers such that $|tn| \downarrow 0$ and $|\phi(tn)| = 1$ for any n. As $\nu 0$ is $k=0 \nu k$, then (16.9) is equivalent 3 always finite, this in turn is equivalent to (x 2 \wedge 1) $\nu(dx) < \infty$. Let fy, n = 1 if there exists an open path of length at most n that connects 0 to y, and fy, n = 0 otherwise. Theorem 13.34 Let E be Polish and let μ , μ 1, μ 2, . . , Ak with kl=1 Al = E, for i1, . Define two measures μ and ν on (Ω , A) by + , Ei μ : Ej \rightarrow P[Ei]. \blacklozenge Remark 14.34 The procedure can be extended to n-stage experiments. Lemma 18.2 For any x \in E, there exists an nx \in N with pndx (x, x) > 0 for all $n \ge nx$. Therefore, $G \in M \le \{\varphi 0 \text{ satisfies an LDP with rate function } I \varphi$. It is enough to show that $\mu 1$ (C) = $\mu 2$ (C) for all closed sets $C \subset E$ as the closed sets form a \cap -stable generator of the Borel σ -algebra that contains E. For general f, apply the usual approximation argument as in Theorem 14.19. Again, after that, the three blocks consisting of Chaps. Define An = An (γ) := max |Xk2-n - X(k-1)2-n|, k \in {1, . Proof Clearly, Rd is Polish and λ is locally finite. Since there are uncountably many A \in A in general, we could not simply unite all the exceptional 204 8 Conditional Expectations sets for any A. R R In particular, if X : (Ω , A) \rightarrow R, B(R) is measurable, then in a canonical way X is also an R-valued measurable map. Let U-n be the number of upcrossings of X over a.b.,) = (Xn1, Xn2, ..., N), k=1 Infer that $\pi(\{v\}) \ge \infty$)* P π $\tau x1 \ge k$, X0 = x, $Xk = v = \pi(\{x\})$ ux ($\{v\}$), k=1 where ux is the invariant measure defined in Theorem 17.48. Conclude that (Wn)n \in N converges almost surely an in L1 to a random variable W that is exponentially distributed with 17.4 Discrete Markov Chains: Recurrence and Transience 411 parameter 1. Then (Ei)i ∈ I is independent => (2.7) holds for J = I. Now for any i ∈ N, let Di ⊂ Ωi be a countable dense subset and let yi ∈ Di be an arbitrary point. Secure and Transience 411 parameter 1. Then (Ei)i ∈ I is independent => (2.7) holds for J = I. Now for any i ∈ N, let Di ⊂ Ωi be a countable dense subset and let yi ∈ Di be an arbitrary point. irreducible random walk X on G. Using Laplace's expansion formula for the determinant (elimination of rows and columns), we get the recursion $\chi N(x) = -x \chi N - 1$ (x) $-r(1 - r) \chi N - 2$ (x). We make the elementary observation that for all $D \subset R$, $\partial f - 1$ (D) \cup Uf. For any $n \in N$, define $kn = (1 + \epsilon)n! \ge 12(1 + \epsilon)n$. Evidently, a locally Hölder of the determinant (elimination of rows and columns), we get the recursion $\chi N(x) = -x \chi N - 1$ (x) $-r(1 - r) \chi N - 2$ (x). We make the elementary observation that for all $D \subset R$, $\partial f - 1$ (D) \cup Uf. For any $n \in N$, define $kn = (1 + \epsilon)n! \ge 12(1 + \epsilon)n!$. y -continuous map is Hölder-y -continuous at every point. (sup f ([ti-1, tin)) 1[ti-1 i i=1 As t n+1 is a refinement of t n, we have $gn \le gn+1 \le hn+1 \le hn + 1 \le hn+1 \le hn+1 \le hn + 1 \le hn+1 \le hn+1$ Variables 49 (vii) Let $\mu \in \mathbb{R}$, $\sigma 2 > 0$ and let X be a real random variable with 1 $P[X \le x] = \sqrt{2\pi\sigma^2}$ (t - μ)2 dt exp - $2\sigma 2 - \infty x$ for $x \in \mathbb{R}$. In particular, $C0 := {\text{Re}(f) : f \in \mathbb{C}} \subset \mathbb{C}$ is a real algebra that, by assumption, separates points and contains the constant functions. $j \in J \neq 2\pi\sigma^2$ (t - μ)2 dt exp - $2\sigma^2 - \infty x$ for $x \in \mathbb{R}$. In particular, $C0 := {\text{Re}(f) : f \in \mathbb{C}} \subset \mathbb{C}$ is a real algebra that, by assumption, separates points and contains the constant functions. such a product formula for Aj from an \circ -stable generator of Aj only. Indeed, for any σ -finite measure μ , we have $\mu(F^n) < \infty$ for all $n \in N$. Formally, the X1, X2, Now assume that (NI, I \in I) fulfills (P1)-(P5). Thus, any μ can be regarded as a measure on E . Klenke, Probability Theory, Universitext, (17.1) 391 392 17 Markov Chains In fact, (17.1) clearly implies the Markov property. \bullet Alternative Solution A different approach to solving the problem of Example 19.32 is to use linear algebra instead of network reduction. (6.4) Define ge = $2g\epsilon/2$. The individual coordinates X(1), . Now let τ n := $2-n 2n \tau + 1!$ for n \in N. Since usually tightness is easier to check, we have a powerful tool for

showing the existence of accumulation points. Let Y0 = 0. $x \in E$ Then: (i) X is ergodic (on $(\Omega, A, P\pi)$). Show that $P[X \in Ht] = 1$, where $Ht = \{x \in Rd : |x, t^* \in 2\pi Z\} = y + z \cdot (2\pi t/t22) : z \in Z, y \in Rd with)y, t^* = 0$., $tN \leq 1$, $y \in Rd with)y, t^* = 0$., $tN \leq 1$, $y \in Rd with)y, t^* = 0$., $tN \leq 1$, $y \in Rd with)y, t^* = 0$. Karhunen-Loève expansion. A set $A \in 2\Omega$ is called $\mu *$ -measurable if $\mu * (A \cap E) = \mu * (E)$ for any $E \in 2\Omega$. Since Pp [DLi] > 0, and since N = m almost surely, we have NLi = m almost surely for i = 0, 1. It is well-known that exp(z1 + z2) = $(E \cap E) = \mu * (E)$ for any $E \in 2\Omega$. Since Pp [DLi] > 0, and since N = m almost surely for i = 0, 1. It is well-known that exp(z1 + z2) = $(E \cap E) = \mu * (E)$ for any $E \in 2\Omega$. Switzerland AG 2020 A. Now assume the gamble is played again and again. Let $A = \sigma (\{\omega\} : \omega \in \Omega \setminus \{\omega\})$. In I = 1/(3 - 2s) 256 11Martingale Convergence Theorems and Their Applications so as to get $\psi_n(s) = E[s Zn] = \text{letting } s = n e^{-\lambda/2}$, (2-2n)s+2n-1. 15.1 Separating Classes of Functions.. To this end, We use this in order to show that br, $p \lor v$ f for $k \in N$, we compute $-(\{k\}) = r - 1$ br, $p \nmid r \lor 0$ $(1 - p) \not_n(1 - p) \dots$ " This has been shown in the discussion above. Let F = 1B. The following theorem gives two simple procedures for calculating the characteristic functions of compound distributions. For $x \in E$, define $\rho A_{,\mathcal{E}}(x) = 1 - \phi \varepsilon - 1 d(x, A)$. (1A f) du We say that f $\mu := \nu$ has density f with respect to μ . assumption, p(x0, y) > 0 now implies $y \in E1$ and $y \in E$ i (for all $n \in N$ and Inductively, we get that pnd+i(x, y) > 0 implies $y \in Ei \cap Ei = 0$, A = 0, Z = 0 be metric spaces and let $f: \Omega \to \Omega \to 0$ be metric spaces and let $f: \Omega \to \Omega \to 0$ be metric spaces and let $f: \Omega \to \Omega \to 0$ be metric spaces and let $f: \Omega \to \Omega \to 0$ be metric spaces and let $f: \Omega \to 0$ be metric spaces and let $f: \Omega \to \Omega \to 0$ be metric spaces and let $f: \Omega \to 0$ be metric spac $P[Xn = xn X0 = x, X1 = x1, .n \rightarrow \infty n \rightarrow \infty (ii)]$ If $\phi n \rightarrow f$ pointwise for some $f: Rd \rightarrow C$ that is partially continuous at 0, then there exists a probability measure Q such that $\phi Q = f$ and Q = w-lim Pn. (ii) Since f + f - = |f|, Lemma 4.6(iii) yields $f + d\mu + f - d\mu$ $f d\mu = f + d\mu - f - d\mu \le f + f - d\mu = |f| d\mu$, AN $\in \mathbb{Z} \mathbb{R}$ n=1 and similarly Z*E R. In fact, $C(x,y) C(x) \pi(\{x\}) p(x, y) = C(x)$ for all $x, y \in E$, then X is reversible with respect to $C(x, y) = C(y, x) = C(y, x) = C(y, x) = r(\{y\}) p(y, x)$. For $x, y \in E$, let) * F(x, y) := Px there is an $n \ge 1$ with Xn = y be the probability of ever going from x to y. 1 2 1 2 $\Gamma\theta, r * \Gamma\theta, s = \Gamma\theta, r + s$ for $\theta, r, s > 0$. Definition 4.7 (Integral of measurable functions) A measurable function $f: 3 \Omega \rightarrow R$ is called μ -integrable if $|f| d\mu < \infty$. Indeed, if $(xn)n \in N$ does not have a limit point, then by the Bolzano-Weierstraß theorem, $\#\{n \in N : xn \in [-K, K]\} < \infty$ for every K > 0. 7.2 Inequalities and the Fischer-Riesz Theorem 165 Now let $p \in [1, \infty)$. That is, $n \rightarrow \infty n |t n| := max\{tin - ti - 1 : i = 1, . Note$ that X is a (bounded) martingale. In the central limit theorem (CLT), we study the size and shape of the typical fluctuations around $n \cdot E[X1]$ in the case where the Xi have a finite variance. Now $\mu(\{d(fn, f) > \epsilon/2\}) + \mu(\{d(fn, f) > \epsilon/2\})$ $= \pi 1$. Find an example of a class of sets E that is not \cap -stable and such that σ (E) = δ (E). n k n-k n k=0 418 17 Markov Chains Note that in the last step, we used a simple combinatorial identity that follows, e.g., by the convolution formula (bn,p * bn,p)({n}) = b2n,p({n}), (ii) Let Ω be a finite nonempty set. For $f \in L2$ ([0, 1]), define I (f) := $\infty \xi n$)f, bn *. Evidently, $\{Xn = 1 \text{ infinitely often}\} = \{\sup S = \sup U\}$ and $\{Xn = 0 \text{ infinitely often}\} = s \text{ and } \sup R = Tr$, we thus have $P[B c] = \{\sup S = \sup U\}$. Thus $A\varepsilon$ is countable. Since $f + \leq g + 4\mu$. (i) For any $A \in \sigma$ (A) and $\varepsilon > 0$, there exist mutually disjoint sets A1, A2, Define $qk*0 = 1\{0\}$ (k) and qk*n = k *(n-1) qk-1 ql for $n \in N l=0$ as the n-fold convolutions of q. Similarly, we get $F(x) = (-x)-\alpha F(-1)$ for x < 0 (with the same $\alpha \in (0, 2)$ since it is determined by the sequence (an $n \in N$). dy fX (x) (8.17) Indeed, by Fubini's theorem (Theorem 14.19), the map $x \to measurable$ for all $B \in B(R)$ and for $A, B \in C(0, 2)$ since it is determined by the sequence (an $n \in N$). B(R), we have $P[X \in dx] \text{ ff } |X(x, y) \lambda(dy) \text{ A B } P[X \in dx] \text{ ff } |X(x, y) \lambda(dy) \text{ is } f(x, y) \lambda(dy) \text{ is } f(x, y) \lambda(dy) \text{ is } f(x, y) \lambda(dy) \text{ B } \lambda(dx) - 1 \text{ 3 f } (x, y)
\lambda(dy) \text{ B } \lambda(dx) - 1 \text{ 3 f } (x, y) \lambda(dy) \text{ B } \lambda(dx) - 1 \text{ 3 f } (x, y) \lambda(dy) \text{ B } \lambda(dx) - 1 \text{ 3 f } (x, y) \lambda(dy) \text{ B } \lambda(dx) - 1 \text{ 3 f } (x, y) \lambda(dy) \text{ B } \lambda(dx) - 1 \text{ 3 f } (x, y) \lambda(dy) \text{ B } \lambda(dx) - 1$ reversible with respect to π . Let A1 , A2 \subset E be open and let C \in C with C \subset A1 \cup A2 . Indeed, Lemma 1.52 would then imply that $\mu *$ is a measure on the σ -algebra of $\mu *$ -measurable sets and the restricted measure $\mu := \mu *$ fulfills E $\mu(A) = \mu * (A) = \beta(A)$ for all open A. At the second stage, i.i.d. random variables with distribution Ξ are implemented. n $n \rightarrow \infty$ (23.16) Note that, in this inequality, in the infimum we cannot simply replace A \cap En by A. Example 20.8 Let $n \in N \setminus \{1\}$, let $\Omega = Z/(n)$, let $A = 2\Omega$ and let P be the uniform distribution on Ω . i \in I Example 14.2 (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, . Two (or more) \text{ edges with resistances } R1 / . By what we have shown already (with X replaced by <math>|X| \land N$ and with $Y = 0 \in L2$ (i) If $\Omega 1 = \{1, ..., N \in L2$ (i) If $\Omega 1 =$ (Ω, F, P) , and using the elementary inequality a 2 $\leq 2(a - b)2 + 2b2$, a, b $\in R$, we infer ') 2 (*2()* $\leq 4E[X2]$. $n \rightarrow \infty$ (ii) We have $X\tau \wedge n \rightarrow X\tau$ almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fourth moments Ex (Z⁻ tn+s - X\tau almost surely. Only here do we need aperiodicity of p. Hence, for fixed N > 0, we compute the fout Z^{-} sn)4 for s, t \in [0, N]. Since I is a good rate function, the level set K := I - 1 ([0, a]) is compact. Indeed, n-fold divisibility alone does not imply uniqueness of the nth convolution root $\mu * 1/n := \mu n$ or of ϕn , respectively. 2 its PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) l= 1 = kn l= 1 ft (x) x 2 + itx PXn, l (dx) l= 1 = kn l= 1 ft (x) x 2 PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) l= 1 = kn l= 1 ft (x) x 2 PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) l= 1 = kn l= 1 ft (x) x 2 PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) - 1 = l = 1$ ft (x) x 2 + itx PXn, l (dx) = itE[Xn, l] = 0, since the array kn $\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn (h, h) = itE[Xn, l] = 0, since the array kn <math>\phi n, l(t) = itE[Xn, l] = 0, since the array kn (h, h) = itE[Xn, l] = 0, since the array kn (h, h)$ 362 15 Characteristic Functions and the Central Limit Theorem = ft dvn n $\rightarrow \infty - \rightarrow -$ t2 (by Lemma 15.50). By the monotonicity principle, we thus have Reff (0 $\leftrightarrow \infty$) \geq Reff \blacklozenge Example 19.29 Let (E, K) be an arbitrary connected subgraph of the square lattice (Z2 , L2). The connection with discrete potential theory will be investigated later, in Chap Theorem 14.12 For any $i \in I$, let $Ei \subset Ai$ be a generator of Ai. Indeed, for $\varepsilon > 0$ and $K = [-C/\varepsilon, C/\varepsilon]$, by Markov's inequality, PXi ($R \setminus K$) = P[|Xi| > C/\varepsilon] $\leq \varepsilon$. The Gibbs sampler is a more specific algorithm often helpful in statistical mechanics. More precisely, for any $f \in L2$ ([0, 1], λ), there exist uniquely defined square summable sequences (an) $n \in N$ and (bn)n \in N0 such that 2 2 f = ha,b. 25.3). The distribution Muln,p on m N0 is called multinomial distribution with parameters n and p. In Theorem 14.50, for every x \in Rd, we constructed a d [0, ∞) d \otimes [0, ∞) with measure Px on (R), B(R) Px \circ (X0, Xt1, Let h \in L1 (µ) with h > 0 a.e. Let ε > 0 and let gc/3 be an ε /3-bound for F (as in (6.5)). Thus, in this case (Ω , A, P, τr) is not ergodic. Let F τn be the σ -algebra 506 20 Ergodic Theory of τn -past. Klenke, Probability Theory, Universitext, 1 2 1 Basic Measure Theory Definition 1.2 (σ -algebra if it fulfills the following three conditions: (i) $\Omega \in A$., Xn,kn be real random variables. For example, the sufficient criterion of absolute summability of coefficients (An) fails (see Exercise 21.5.5). We summarize the discussion in a theorem due to Chung and Fuchs [27]. 7.1 Definitions We always assume that (Ω, A, μ) is a σ -finite measure space. 21.2 for a computer simulation of Xn , n = 0, 1, 2, 3, 10. However, as in the proof of existence of Brownian motion, second moments are not enough; rather we need fourth moments in order that we can choose $\beta > 0$. Hence, it suffices to show that this condition follows from the assumption E[X2] < ∞ . \blacklozenge Example 11.17 (Radon-Nikodym theorem) With the aid of the martingale
convergence theorem, we give an alternative proof of the Radon-Nikodym theorem (Corollary 7.34). \in A0. Note that X is a Markov chain only given W; that is, under the probability measure P[X $\in \cdot |W$]. Remark 19.9 If X is reversible with respect to π , then π is an invariant measure for X since $\pi(\{x\})$ p(x, y) = $\pi(\{x\})$. Hence, let X be uniformly distributed on [0, 1]. Exercise 7.2.1 Show Hölder's inequality by applying Jensen's inequality to the function of Example 7.13. (i) Case M(E). \bullet 10.1 Doob Decomposition and Square Variation 231 Example 10.6 Let Y1, Y2, \bullet Exercise 8.3.5 (Borel's paradox) Consider the Earth as a ball (as widely accepted nowadays). + km = n, we have n k P[Y = k] = Muln, p({k}) := p . 18.1 The left Markov chain is periodic with period 2, and the right Markov chain is aperiodic. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar methodology now known or hereafter developed. It is a tough book so if you are studying probability theory (and measure theory) for the first time don't read this book alone. The theorem on the tilted LDP yields that the sequence of Boltzmann distributions β (μ) $n \in N$ satisfies an LDP with rate n and rate function I (x) = $\beta \cdot F \beta$ (x) $-\beta$ inf $y \in M1$ (Σ) βF (y). That is, in each time step, a coin flip decides whether X makes a jump according to the matrix p^{2} or p. k=1 (' E (Sk + c)(Sn - Sk) + (Sn - Sk)(2 + c)(Sn - Sk)(2 + c)(Sn - Sk) + (Sn - Sk)(Sn - Sk) + (Sn - Sk)(Sn - Sk) + (Sn - Sk)(Sn - Sk) +k=1 + (E(Sk+c)(2) + (k+c)(2) + (k+c)(2)), we have $n \to \infty$ (Xtn1, We make the following simple observation. Then C is a separating family for Mf (E). Let $D = (D_1)(e^{-1}) + (k+c)(2)$, we have $n \to \infty$ (Xtn1, We make the following simple observation. Then C is a separating family for Mf (E). Let $D = (D_1)(e^{-1}) + (k+c)(2)$, we have $n \to \infty$ (Xtn1, We make the following simple observation. Then C is a separating family for Mf (E). and $p \in [1, \infty)$. Evidently, a necessary condition is that (μ n (E)) $n \in N$ is bounded. l=1 Lemma 15.48 For every $t \in R$, we have $ft \in Cb$ (R). Thus, for large n, the dependence of An (ϕ) on the first l coordinates is negligible. The process N[°] fulfills (i) and (ii), but not (iii). the event B c where infinitely many balls of each color are drawn. By (iv), $(X - a) + = Y \lor 0$ is also a submartingale. This is more than just coincidence. Then, for every $z \in E$, $\mu x p(\{z\}) = y \in E \mu x (\{y\}) p(y, z) = \infty$ n=0 y $\in E \mu x (\{y\}) p(y, z) = \infty$ n=0 ndefine independence of larger families of events, we have to request the validity of product formulas, such as (2.2) and (2.3), not only for pairs and triples but for all finite subfamilies of events. Using the Markov property (third and fifth equalities in the following equation) and the induction hypothesis (fourth equality), we get ' (Ex f (Xt + s)t ≥ 0 Fs ') * (= Ex Ex f (Xt + s)t ≥ 0 Ftn + s Fs (') * = Ex PXtn + s 1B1 (Xt1 + s) · · · 1Bn (Xtn + 1 - tn ∈ Bn+1 B1 (Xt1 + s) · · · 1Bn (Xtn + 1) + s ∈ Bn+1 Ftn + s 1B1 (Xt1 + s) · · · 1Bn (Xtn + 1) + s ∈ Bn+1 Ftn + s 1B1 (Xt1 + s) · · · 1Bn (Xtn + 1) + s ∈ Bn+1 Ftn + Up to a factor, we would thus get (17.18) without using the multidimensional local central limit theorem. By (iv) and (v), φ is differentiable in I \circ \ A with derivative D + φ. The ideal tools for the treatment of central limit theorems are so-called characteristic functions; that is, Fourier transforms of probability measures. Exercise 17.1.1 Let I \subset R and let $X = (Xt) t \in I$ be a stochastic process. 3 Exercise 4.2.2 Let f1, f2, $0 \neq 0$ be closed under addition and assume $0 \in I$. Then Dom(μ) = \emptyset if and only if μ is stable (in the broader sense). Let $Sn * := X1 + ... + Xn \sqrt{.n} + ... + ... + Xn \sqrt{.n} + ...$ walk on Z that jumps one step to the right with probability r and one step to the left with probability 1 - r (for some $r \in (0, 1)$)., k, pairwise distinct j1, k=0 Proof Define n-1 p 1 Yn := Xk - E[X0 | I] n for every $n \in N$. We first define in metric spaces almost sure convergence and convergence in measure and then compare both concepts. This shows that the concepts formed in Sect. For $x \in E$ and r > 0, denote by Br (x) = { $y \in E : d(x, y) < r$ } the ball with radius r centered at x. 21.6 The Space C([0, ∞)). Finite measures on Polish spaces are Radon measures. Then (Xn)n \in N fulfills the weak law of large numbers. Hence we can solve (9.3) and get HT := $\int + - \int VT - VT dV = VT - VT dV$. $XT + = XT - \beta$, else, and VT - 1 = VT - HT(XT - XT - 1) = VT - HT(XT - XT - 1). The indicator function $1A : \Omega \to \{0, 1\}$ is $A - 2\{0, 1\}$. infinitely n divisible. 6.2 Uniform Integrability 159 subsequence Convergence almost everywhere rm i fo i t y un rabil eg int e nc ue seq sub u int nifo eg rm rab ilit y Stochastic convergence Fig. Theorem 21.48 There is a continuous version of the Markov process Y with transition kernels (kt)t ≥0 given by (21.45). Show that the limit $\lim n \to \infty$ an /n exists and that $\lim n \to \infty$ 1 1 an = inf an . Since (C([0, 1]), $\cdot \infty$) is complete, it suffices to show that U \in U back that U \in U has Lebesgue measure at least $(1 - \varepsilon)\lambda(W)$. Exercise 13.2.9 Show the implication "(vi) \Rightarrow (iv)" of Theorem 13.16 directly. Define ∞ (x - an)+ H (x) = for any x ≥ 0 and F (x) = $-\#\{n \in N : xn \in [0, x]\}$ for x < 0, then $\mu = \mu F$. Furthermore, let fn be constant to the right of n and for x < 0, define fn (x) = fn (-x). Takeaways In many situations, stochastic processes in continuous time are constructed as limits of simpler processes. The arguments we gave there were rather abstract. 3 Then the map $F: E \to R$, $x \to f(\omega, x) \mu(d\omega)$ is continuous at x0. Then there is a set $\Omega + \epsilon A$ with $\phi(A) \ge 0$ for all $A \in A$, $A \subset \Omega - := \Omega \setminus \Omega + \alpha$. $\in n \rightarrow \infty$ M(E) be measures such that $\mu n \rightarrow \mu$ vaguely. By the usual exhaustion arguments, we can restrict ourselves to the case where μ and ν are finite. Now let s, t \in I with $|s - t| \geq$. Hence, for $n \geq n0$, $\mu n (A) - \mu(A) \leq \mu n (\{N + 1, N + 2, . Then P equals the voltage at point x: P = u(x). Theorem (1956))$ Let (E, d) be a metric space and $F \subset M \leq 1$ (E). 20.2 Ergodic Theorems 497 Hence, if we let T be the tail σ -algebra of (Xn) n \in N (see Definition 2.34), then I $\subset T = \infty \sigma$ (Xn , Xn+1, . Now let H be progressively measurable, and assume E 0 Ht2 dt $< \infty$. The paths are Hölder continuous of any order less than 1/2, but almost surely they are not Hölder continuous of any order larger than 1/2 at any point. In many situations it is desirable to have a coupling with additional properties like all the mass lies above the diagonal. " = " Let $Nn = \{f \ge n1\}$, $n \in N$. Hence (iii) follows. \blacklozenge Let $X = (Xn)n \in N$ is an array of random variables. Hence X is a Markov chain on N0 with transition matrix p. be sets. For larger values of ε , we have $|\lambda\varepsilon,1| > |\lambda\varepsilon,N/2|$. The existence of the Poisson process has not yet been shown. Now assume (ii). (ii) Let $\Omega = \{1,2\}$ and $E = \{\{1,2\}, \tau xk \text{ is the kth entrance time of } X$ for x. Assume that in the beginning there are R red balls and S black balls in the urn. If f is smooth in some sense, then the usual numerical procedures yield better orders of convergence. Theorem 16.13 Let (Xn, 1; l = 1, $\phi\mu$ is called the characteristic function of μ . Such a process X is called a Feller process. \blacklozenge Lemma 1.31 (Properties of contents) Let A be a semiring and let μ be a content on A. (5.14) $n \rightarrow \infty$ For $\delta > 0$ and $n \in 1$ N, define $A\delta n := max\{|Sk| : k \le kn\} > \delta l(kn)$. Then
)* FY (x) = P Xi \le x for all i = 1, . We construct explicitly a candidate X for a Markov process with Q-matrix q. $k \to \infty$ Reflection Check that in the above proof, in general, we do not have F (q) = F[°] (q) for all q $\in Q$. \diamond 294 13 Convergence of Measures Proof (of Theorem 13.29(i) for the case E = R) Assume F is tight and (µn) $n \in N$ is a sequence in F with distribution functions Fn : $x \to \mu N$ (($-\infty, x$])., T, define Xn = (1 + b) Xn-1, if Dn = +1, (1 + a) Xn-1, if Dn = -1. N > 0 there are numbers C, α , β > 0 such that, for all s, t \in [0, N] and every i \in I, we have) * E |Xsi - Xti | $\alpha \leq C$ |s - t| β +1.7 Lp -Spaces and the Radon-Nikodym Theorem 166 Example 7.5 (i) The convex subsets of R are the intervals. , An are pairwise disjoint; hence We thus obtain *) Var[Sn] + c2 = E (Sn + c)2. Then f $\leq \infty \cdot 1$ N and n1N $\uparrow \infty \cdot 1$ N Concluding, we have $z, w^* = 0$ for all $w \in W$ and thus $z \in W \perp$. be independent random variables with E[Xn] = 0 for any $n \in N$ and $v := \sup\{Var[Xn] : n \in N\} < \infty$. \bullet Lemma 9.18 Let $I \subset [0, \infty)$ be closed under addition and let σ and τ be stopping times. Hence ∞ n=1 An $=\infty$ (A1 \cap An $) = n=2 \infty$ A1 $(A1 \setminus An) = A1 \setminus n=2$ (iii) Assume that A1, A2, . $k=m+1 = 1[0,t] 2 = t < \infty$, we have $Xtn \in L2$ (P) and *) lim sup E (Xtm - Xtn) 2 = 0. Hence, for all $m \in N$ (with $n0 = "log m/log \alpha #$), $n: kn \ge m kn-2 \le 4 \infty \alpha - 2n = 4 \alpha - 2n0$ ($1 - \alpha - 2$) $-1 \le 4(1 - \alpha - 2) - 1 \le 4(1 - \alpha - 2)$ derive a version of Cramér's theorem for Rd -valued random variables taking only finitely many different values. For x > 0, we have P1 [e-t Xt > x] = P[Xt > et x] = (1 - e-t) et x! $t \rightarrow \infty \rightarrow e-x$., E12 from Theorem 1.23 (with n = 1) is a generator for B(R). For $i \in N$, let di be a complete metric that induces $\tau i \cdot B2$ A2 C Evidently, B1 \subset A1 and B2 \subset A2 $and \propto n=1$ An = A, and • An \downarrow A and say that (An)n \in N decreases to A if A1 \supset A2 \supset A3 \supset . " \Rightarrow " Let J \subset I be finite. If C(x, y) = 0, then we could just as well assume that there is no wire connecting x and y. Remark 9.38 Clearly, H \cdot X is adapted to F. Exercise 15.6.1 Let $\mu \in \mathbb{R}^d$, let C be a symmetric positive semidefinite real d × d matrix and let X \sim Nµ,C (in the sense of Remark 15.56). ., we obtain $[\omega 1] \cap Bn = \emptyset$ for all $n \in N$. , N - 1) $\leq C \sigma \cos N$ for every $n \in N$. The remaining random graph almost surely contains a (unique) infinite connected component if p is larger than a critical 1 value pc. Hence, let wi - $\in (0, 1)$ and wi + := 1 - wi - for i $\in Z$. # {1, 3} 2 = . n n $\xi n (X)$:= i=1 $n \rightarrow \infty$ Note that by the law of large numbers, P-almost surely $\xi_n(X) \rightarrow \mu$. (iii) Show that the space Cc ([0, ∞)) of continuous functions with compact support, equipped with the supremum norm, is separate points if for any two points x, y \in E with x = y, there is an $f \in C$ with f(x) = f(y). Assume that we have $n \rightarrow \infty \phi n(t) - \rightarrow 1$ for t in a neighborhood of 0. Chapter 9 Martingales One of the most important concepts of modern probability theory is the martingale, which formalizes the notion of a fair game. In this case, we say that p is translation invariant. (i) ν is called absolutely continuous with respect to μ (symbolically $\nu 0 \mu$) if $\nu(A) = 0$ for all $A \in A$ with $\mu(A) = 0$. Define $\lambda := \lim \sup \varepsilon - 1$ P[N $\varepsilon \ge 2$]. By the translation invariance of the lattice, we have θ (p) = P[#C p (y) = ∞] for any $y \in \mathbb{Z}d$. 2 $\Lambda * (z) = However$, this is the rate function from Theorem 23.1. \blacklozenge Takeaways For random variables with exponential moments, in the weak law of large numbers, the probability for large deviations decays exponentially fast. Further, let $a \ge 0$ and $b \ge 0$. Compact sets are closed. A class of sets $\tau \subset 2\Omega$ is called a topology on Ω if it has the following three properties: (i) \emptyset , $\Omega \in \tau$. A continuous version of X can be obtained via the Kolmogorov-Chentsov theorem (Theorem 21.6). By Kolmogorov's 0-1 law (Theorem 2.37), the tail σ -algebra T = $n \in N \sigma$ (Y, $m \ge n$) is P-trivial. Assume that (PXi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, $i \in I$ (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$) and (PYi, i \in I (PYi, $i \in I$ independent. Hence X have P Xt = X (ii) Let $\varepsilon > 0$ and choose $n \in N$ large enough that (see (21.6)) P[Bn] $\leq C 2 - (\beta - \alpha \gamma) n < \varepsilon \in M1$ (R) with $\mu n \rightarrow \mu$ weakly. M(E) Here we need the slightly larger space in order to define random measures again yield random measures. 13.2 Weak and Vague Convergence . (M - I) = N! N! = 0 = 0 (iii) Let Y be a random variable with values in [0, 1]. (i) Show that $\infty = 1$ Var[Xi] $< \infty$ and E[X1] = 0. We define $\beta(A) := 0$ (iii) Let Y be a random variable with values in [0, 1]. (i) Show that $\infty = 1$ Var[Xi] $< \infty$ and E[X1] = 0. We define $\beta(A) := 0$ (iii) Let Y be a random variable with values in [0, 1]. (ii) Show that $\infty = 1$ Var[Xi] $< \infty$ and E[X1] = 0. We define $\beta(A) := 0$ (iii) Let Y be a random variable with values in [0, 1]. (i) Show that $\infty = 1$ Var[Xi] $< \infty$ and E[X1] = 0. We define $\beta(A) := 0$ (iii) Let Y be a random variable with values in [0, 1]. (ii) Show that $\infty = 1$ Var[Xi] $< \infty$ and E[X1] = 0. We define $\beta(A) := 0$ (iii) Let Y be a random variable with values in [0, 1]. (ii) Show that $\infty = 1$ Var[Xi] $< \infty$ and E[X1] = 0. We define $\beta(A) := 0$ (iii) Let Y be a random variable with values in [0, 1]. sup $\alpha(C) : C \in C$ with $C \subset A$ for $A \subset E$ open and $\mu * (G) := \inf \beta(A) : A \supset G$ is open for $G \in 2E$. By Theorem 6.18(ii) (with $G = \{f\}$), we obtain that the family $(f - fnk \mid -g) + d\mu < \epsilon$. By the Heine-Borel theorem, a subset of Rd is compact if and only if it is bounded and closed. The proof is similar to the proof of Theorem 6.25. By the strong Markov property, (Bt) $t \ge 0 := (B\tau + t)t \ge 0$ is a Brownian motion started at a and is independent of $F\tau$. In particular, $|gn| \le h$ almost everywhere for all $n \in N$. By Theorem 2.13(ii), we can even assume that K is finite. For $\varepsilon > 0$, by the convexity, the map $x \to gx$ ($x + \varepsilon$) is monotone increasing and is continuous by (i). This array is a null array if and only if (16.4) holds. 15. Let Bb (E) = B \in B(E) : B is relatively compact be the system of bounded Borel sets and M(E) the space of Radon measures on E (see Definition 13.3). N – 2}, if j = i \in {0, N}, else. Thus σ Xt, t \in [0, ∞) \subset B(Ω , d). In this case, the claim VT is called replicable and the strategy H is called a hedging strategy, or briefly a hedge. $\geq E X1/p + Y 1/p \blacklozenge$ Before we present Hölder's inequality and Minkowski's inequality, we need a preparatory lemma. The gambling strategy H := $2n-1 1 \{D1 = D2 = ... = D n-1 = -1\}$ for $n \in N$ and H0 = 1 is predictable and locally bounded. Indeed, for $k \in N$ and mutually distinct p1 zy xzz zyy zy zzz zx zx zzz zx zzz zx zzz zx zzz zx Fig. In Chap. Analogously, we find a k subsequence (n2k) k N of (n1k) process and characterize it in terms of its Laplace transform. On the other hand, f is Lebesgue integrable with integral [0,1] f $d\lambda = 0$ because Q \cap [0, 1] is a null set. Now integrate over [0, t] and sum up. 4.3 Lebesgue Integral Versus Riemann Integral 111 3 ∞ 3 k $\rightarrow\infty$ Since 2-k g(ϵ) $\rightarrow 0$, we get 0 g ϵ (t) dt = f ϵ d μ . We show the validity of (2.8) with J replaced by J^{*}. Thus we get the upper bound lim sup $n \rightarrow \infty$ 1 log P[Sn ≥ 0] $\leq \log = -\Lambda * (0)$. Replace the series on top, bottom and right by edges with resistance 2 (right in Fig. Hence, for every $n \in N$ there is an in $\in \{1, ..., Am \in A, \text{then } (4.1)$ is said to be a normal representation of f. Show that (15.4) holds., xtn+1) (with B1, ..., Am \in A, \text{then } (4.1) is said to be a normal representation of f. Show that (15.4) holds. Sn = X1 +. Case 2. " \leftarrow " Now assume that (i) hold. If ν is totally continuous with respect to μ , then ν 0 μ . Hence the assumption was false. (Note in particular that every metric space is a T3 1 -space.) 2 Show that
σ (Cb (Ω)) = B(Ω); that is, the Borel σ -algebra is generated by the bounded continuous functions $\Omega \rightarrow R$. For example, choose -1/2Hn (t) = n n hn (F (Xi) - t) - gn (t), i=1 where hn is a suitable smoothed version of $1(-\infty,0]$, for example, hn (s) = $31 1 - (s/\epsilon n \vee 0) \wedge 1$ for some sequence $\epsilon n \downarrow 0$, and gn (t) := 0 hn (t - u) du. Each Bx is a union of elements of U each of which is then also relatively compact. 19.2)., DT be i.i.d. Radp random variables (that is, P[D1 = 1] = 1 - P[D1 = 1-1 = p). Choose n0 \in N large enough that |fn (ti) - f (ti)| $\leq \varepsilon$ for all i = 1, . While Strassen's theorem yields the existence of an abstract coupling, in many examples a natural coupling can be established and used as a tool for proving, e.g., stochastic orders. (5.2) 5.1 Moments 117 Proof + , m n α i Xi , e + β j Yj Cov d + j = 1 i= 1 = E + m α i (Xi - E[Xi]) $n_j = 1 = 1 = n m$, $\beta_j (Y_j - E[Y_j]) + \alpha_i \beta_j E(X_i - E[Y_j]) = 1 = n m$ $\alpha_i \beta_j Cov[X_i, Y_j]$. Further, assume that the maps Fn can be chosen to be almost surely monotone increasing. By passing to f/f p and g/gq, we may assume that f p = gq = 1. Check that if E[X_i] = 0 for all i \in N, then instead of independence of T, it is enough to postulate: {T ≤ n} is independent of Xn+1, Xn+2, . n Finally, let Ntn := 2k=1 Xn (k). Then f (x) - f (y) ≤ K d(x, y) ∧ 2 f ∞ for all x, y ∈ E., 6} such that Formally, we assume that there are sets A, A = A^{*} × {1, . Then ν has the density f (x) = $\sqrt{2} 2\pi ex/2$ with respect to μ . To do a reverse number lookup, choose a site that offers the service, such as WhitePages, navigate to the phone lookup section and enter the number. Note that p is reversible (see Sect. I am especially indebted to my wife Katrin for proofreading the English manuscript and for her patience and support. Evidently, ∞ P[Xn - Xn-1 ∞ > 2-n/4] < ∞ ; hence, by the Borel-Cantelli n=1 lemma, '; P :Xn - Xn-1 : ∞ > 2-n/4 (only finitely often = 1. Show that (Xn)n \in N0 is i.i.d. given Z and Xi ~ BerZ for all i \in N0. Now Yn - Y0 = (1 · Y)n = (H · Y)n + ((1 - H)· Y)n : hence *) *) E[Yn - Y0] \geq E (H · Y)n \geq (H · Y)n \geq (H · Y)n \geq (E [ϕ (Xt *)+ Ft] = E ϕ (Xt *) + Ft] = E ϕ (Xt *) + (1 - H)· Y)n : hence *) *) E[Yn - Y0] \geq E (H · Y)n \geq ∞ . Example 21.29 (Stochastic integral à la Paley-Wiener) Assume that (ξ_n) $n \in N$ is an i.i.d. sequence of N0,1 -distributed random variables. Definition 2.34 (Tail σ -algebras. Then A = ∞ n=1 (A \cap Bn). Indeed, v lies in every open neighborhood of 0; hence F assumes at v the same value as at 0. Define u(x) = 0 for every $x \in A0$ and u(x) = 1 for every $x \in A1$. A Recall that, for any subset $A \subset \Omega$ of a topological space (Ω, τ), the class τ is A the topology of relatively open sets (in A). Proof Let $B \in B(R)$ and let $\phi : E \to B$ be an isomorphism of measurable spaces. Show this with a different approach by checking condition (i) from Lemma 3.6. & 3.3 Branching Processes Branching processes are models for the random development of the size of a population. Using an argument similar to that invariant events (defined by i.i.d. random variables) have probability either 0 or 1 (see Example 20.26 for a stationary on $(\Omega, A, P\pi)$. The Fourier basis is not too well suited to showing continuity of paths. Under what conditions do we have q = 0, q = 1, or $q \in (0, 1)$? In addition, the classes E4, . Note that the graph HL has no circles. By Theorem 1.4, A is closed under intersections and is hence a semiring. Define $/l-2 l-1 \kappa k$, /k+1. Definition 1.102 (Random variables) Let (Ω , A) be a measurable space and let X : $\Omega \rightarrow \Omega$ be measurable. By 2017, many states in the United States had even made it illegal to print phone books, according to TruthFinder. (ii) A triple (Ω , A, μ) is called a measurable space and if μ is a measurable space and let X : $\Omega \rightarrow \Omega$ be measurable. By 2017, many states in the United States had even made it illegal to print phone books, according to TruthFinder. (ii) A triple (Ω , A, μ) is called a measurable space and if μ is a measurable space and let X : $\Omega \rightarrow \Omega$ be measurable. Transience of Random Walks 417 This implies for all $L \in N$ GN $(0, y) \ge 1$ GN $(0, y) \ge 1$ GN $(0, y) \ge 1$ H $|y| \le L = N 1$ k p $(0, y) \ge 1$ GN $(0, y) \ge 1$ $\{-\infty\}$ be the time of last entrance in x before time n. 104 4.3 Lebesgue Integral Versus Riemann Integral . \bullet Definition 14.3 (Coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$, $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$, then Xi : $\Omega \to \Omega i$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If $i \in I$ and $\omega \to \omega(i)$ denotes the ith coordinate maps) If (i \in I) denotes the ith coordinate maps) If (i \in I) denotes the ith coordinate maps) If (i \in I) denotes the ith coordinate maps) If (i \in I) denotes the ith coordinate maps) If (sets of finite dyadic rationals Dm = {k2-m, k = 0,more S4nNy rated it it was amazing Nov 16, 2017 Gustavo Ganso rated it really liked it Mar 31, 2021 Tpinetz rated it it was amazing Mar 22, 2020 John rated it really liked it Jan 19, 2013 Philipp rated it as to-read Apr 21, 2013 Du Phan marked it as to-read Apr 21, 2013 Du Phan marked it as to-read May 02, 2013 Harry marked it as to-read Jun 10, 2013 Liam marked it as to-read Jun 13, 2016 sprunghaft marked it as to-read Mar 04, 2017 Ludwig Van marked it as to-read Jun 11, 2018 Evan marked it as to-read Jun 11, 2018 Daniel marked it as to-read Jun 11, 2018 Evan marked it as to-read Jun 12, 2019 Taylor marked it as to-read Jun 12, 2019 Juan Bono is currently reading it Apr 07, 2020 Petra marked it as to-read May 05, 2020 Xiaocan Li marked it as to-read May 08, 2020 Tosiaki is currently reading it May 24, 2020 \otimes 1996-2015, Amazon.com, Inc. Then p g p (x, y) > 0 for every i $\in \Lambda$. Then $X := (Xt) t \ge 0$, $(Px) x \in E$ is a Markov process and ∞ pt $(x, y) := Px [Xt = y] = PT0 [Tt = n] PYx [Yn = y] n=0 = e - \lambda t \infty$ $\lambda n t n n=0 n! pn (x, y)$. 1.2 Set Functions 11 Takeaways σ -algebras are classes of sets that are stable under countable intersections and unions. This power series (in t) converges everywhere (note that as a linear operator, p has finite norm $p \infty \le 1$ to the matrix exponential function $e\lambda tp(x, y)$. , Xn } from Remark 2.24 to compute the Laplace transform LM (t) = n! (t + 1)(t + 2) \cdots (t + n) for t \ge 0. This type of search doesn't deliver consistent results, though it can be useful and delivers some results in map or satellite form. Brief History of Phone BooksThe first printed phone book was handed out in 1878 in New Haven. 18.2 Coupling and Convergence Theorem 439 Property (18.6) says that X visits the Ei one after the other (see Fig. Now $\{\tau \leq t\} \in Ft$ since τ is a stopping time. (ii) Let X and Y be martingales and let a, $b \in R$. Now assume that r is irrational. 2 i Here F \cdot X is the discrete stochastic integral (see Definition 9.37). • 502 20 Ergodic Theory Example 20.18 Let P and Q be probability measures on the measurable space (Ω , A), and let (Ω , A, P, τ) and (Ω , A, Q, τ) be ergodic., n EN = k 1 NEN (Al) m. + Xn, n \Rightarrow N0,1. In order to show (5.12), choose $\tau^- := \min k \in \{1, ..., BtN + x\}$ is continuous and bounded. (8.11) Consider the special
case where F = σ (X) for a random variable X (with values in an arbitrary measurable space (E), E)). If $\beta \leq 1$, then this is the only solution and F β attains its global minimum at m = 0. We pick up this thread again in Sect. However, it is not too hard to show the following theorem, which for the case $\gamma = 1$ is due to Paley, Wiener and Zygmund [126]. 18.5 and 18.6 for computer simulations of equilibrium states and metastable states of the Ising model. The family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changing the distribution of the family X = (Xn)n \in N is called exchangeable if finitely many of the Xi can be permuted without changeable if finitely X = (Xn)n \in N is called exchangeable if finitely X = (Xn)n depends only on the distance (length of the shortest path) from the origin., m = m i=1 $e^{\lambda i} k \lambda i i k!$. The $n \rightarrow \infty$ sequence (Xn) $n \in N$ fulfills the strong law of large numbers if Tn / $n \rightarrow \mu$ a.s. 5.3 Strong Law of Large Numbers 127 ∞) *) * P |Xn | > n \leq E |X1 | < ∞ . , 2n } $\geq 2 - \gamma$ n and Bn := ∞ Am and m=n N := lim sup An = $n \rightarrow \infty \infty$ Bn . In particular, #C p (x) is a random variable for any $x \in Zd$. $e \in A^{\tilde{c}}$ j pe for all $i \in N$, (2.4) $j \in J$ Since this holds for all finite $J \subset N$, the family (Ai)i $\in N$ is independent. Then there exists an $\varepsilon > 0$ and $a^{\tilde{c}}$ nk, f) > ε for all $k \in N$. 24.1). On the other hand, $A = \{\emptyset\}$ and $A = 2\Omega$ are the trivial examples of semirings, rings and σ -rings. Inductively, define stopping times $\tau 1 := \inf\{k \in N : Sk < L + \epsilon\}$ and $\tau n + 1 := \inf\{k > \tau n : Sk < L + \epsilon\}$ for $n \in N$. If, on the other hand, $\lambda(A) = \infty$, then for any L > 0, we have to find a compact $n \to \infty$ set $C \subset A$ with $\lambda(C) > L$. Example 10.2 Let I = N0 or $I = \{0, . Letting \epsilon \downarrow 0 \text{ yields } 0 = P[A](1 - P[A])$. Definition 9.19 Let τ be a stopping time. Furthermore, by (4.7) (with f replaced by fke), we $\mu(\{fke \ge n\}) \le f \in d\mu$ n=1 $\le 2-k \infty$ n=0 $\mu(\{fke > n\}) = 2-k \infty$ n=0 $\mu(\{fke > n\}) \le \alpha ke + 2-k g(e)$. p is the transition matrix of simple (asymmetric) random walk on the discrete torus Z/(N), which with probability r makes a jump to the right and with probability r makes a jump to the left. Thus $\delta = 0$. The details are left as an exercise. For every $n \in N$, let Sn = Xn, 1 + . are real and i.i.d. with $E[Z1 \mid] < \infty$, then $1 \mid n \to \infty$ $Zi \rightarrow E[Z1 \mid] n$ a almost surely. In this we infer P[Z > 0] = 0 and similarly (with -X instead of X) also lim inf $n1 \mid Sn \ge 0$ $n \to \infty \mid n \mid Sn \mid - 0$ a.s. almost surely. In this chapter, we provide the abstract framework for the investigation of convergence of measures. Now $\{(-\infty, r], r \in Q\}$ is a π -system that generates B(R). Let $A - := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n > 0\}$. IAn $(-\infty, r], r \in Q\}$ is a π -system that generates B(R). Let $A - := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1^* = 0$ for some $n < 0\}$ p and $A + := \{X\}n, n+1$ filtration generated by X, not the, possibly larger, filtration generated by D1, . We use this to construct the Metropolis matrix (see [70, 114]). Let $\varepsilon > 0$. Define I = Ik. n \in N Hence also C := max($\mu(E)$, sup{ $\mu(E)$: $n \in N$ }) < ∞ , and we can pass to μ/C and $\mu n / C$. Note that, on the trivial σ -algebra, Dirac measures for different points $\omega \in \Omega$ cannot be distinguished. 5.2). Sn = Proof Without loss of generality, assume E[Xi] = 0 for all $i \in N$ and thus $X1 + \cdots + Xn \cdot 2 \cdot 2 \cdot 2$ Hence the average free energy of a particle is '1 + m 1 + m 1 - m (1 F β (m) = -m2 - hm + β - 1 log + log. Using the moment criterion, here we have shown that also the process of partial sums converges and that the limit is Brownian motion. Be careful, the cases r = 12 and r = 12 are different. This implies $\infty n = 1$ Bn = \emptyset , contradicting the assumption. m=1 Clearly, for $n \ge m$,) * E (Xtm - Xtn)2 = E - n B {k 1[0,t], bk * n C B {l 1[0,t], bk * " \supset " We have to show that open subsets of (Ω , d) are in A := (B(R)) \otimes [0, \infty). Example 1.30(iii).) Let Ω be a countable set, and define A = {A $\subset \Omega$: #A < ∞ or #Ac < ∞ }, $\mu(A) = 0$, ∞ , if A is finite, if A is finite, if A is finite, if A is finite, if A is finite. Definition 13.3 A σ -finite measure μ on (E, E) is called (i) locally finite or a Borel measure if, for any point $x \in E$, there exists an open neighborhood U x such that $\mu(U) < \infty$, (ii) inner regular if $\mu(A) = \sup \mu(K) : K \subset A$ is compact for all $A \in E$, (iv) regular if $\mu(A) = \inf \mu(U) : U \supset A$ is open for all $A \in E$, (iv) regular if μ is inner and outer regular if $\mu(A) = \inf \mu(U) : U \supset A$ is open for all $A \in E$, (iv) regular if $\mu(A) = \inf \mu(U) : U \supset A$ is open for all $A \in E$, (iv) regular if $\mu(A) = \inf \mu(U) : U \supset A$ is open for all $A \in E$, (iv) regular if $\mu(A) = \inf \mu(U) : U \supset A$ is open for all $A \in E$, (iv) regular if $\mu(A) = \inf \mu(A) = \inf \mu($ uniformly continuous on D; hence it can be extended to [0, 1]. (16.33) Theorem 16.29 Let PX be in the domain of attraction of an α -stable distribution (ii) or (iii) of Theorem 16.28 holds), and assume that (an $n \neq \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n
\rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $C = \lim n \rightarrow \infty$ n U (an $) \in (0, \infty)$ and assume that $(0, \infty)$ and $(0, \infty$ ∞ . Example 19.3 Let X be transient and let a \in E be a transient state (that is, a is not absorbing). (ii) In particular, if FA (x, y) > 0 for all x, y \in E \ A. By Remark 8.26, measurability holds for all B \in B(R) and hence κ is identified as a stochastic kernel. In this chapter, we investigate continuity properties of paths of stochastic processes and show how they ensure measurability of some path functionals. We have used the convergence theorems from the last section to show that the integral is continuous or differentiable, respectively, if a regularity assumption is fulfilled. This finishes the proof. If f1 = f2 on A, then f1 = f2. In order to show (2.3), we compute $P[A1 \cap A2 \cap A3] = 33 \#(A^2 \times A^3) \#A^2 = P[Ai]$. (Recall that inf f (A) = inf{f (x) : x \in A}.) If K \subset E is compact and nonempty, then f assumes its infimum on K. Exercise 13.4.2 Show that a family (Xn)n \in N of random variables is exchangeable if and only if, for every choice of natural numbers $1 \le n1 < n2 < n3$. Similarly, choose $C2 \in C$ with $B2 \subset C2 \subset A2$. For s, $t \in [0, 1]$, we have - n n. (i) d := dx = dy for all x, $y \in E$. But modern random number generators produce sequences that for many purposes are close enough to really random sequences.) Define r(i, 0) = 0, r(i, j) = p(i, 1) + . Define $Sn := n \in N0$. 4.1 Construction and Simple Properties 97 3 Remark 4.5 By Lemma 4.3(iii), we have I (f) = f dµ for any f \in E+. Definition 15.54 Let C be a (strictly) positive definite symmetric real d × d matrix and let $\mu \in$ Rd. We first show that A0 is a σ -algebra by checking (i)-(iii) of Definition 1.2: (i) Clearly, $\Omega \in$ A0. In the special case where for every n, the individual ϕ n, l are equal and where $n \rightarrow \infty$ kn $- \rightarrow \infty$, equation (16.4) holds automatically if the product converges to a continuous function. Evidently, #Fn $< \infty$ for all $n \in N$. For any $F \in F$, we C 3 3C thus have E[fF 1F] = F f dP = Q(F) = F f dP = Q(F) = F f dP = Q(F) and thus by the moments of X. In this case, A can be chosen to equal the exchangeable σ -algebra E or the tail- σ -algebra T . \in A $\infty \infty$ Ai , we have $\mu(A) \leq \mu(Ai)$. Hence we define the effective resistance between x0 and x1 as Reff (x0 \leftrightarrow x1) = 1 1 u(x1) - u(x0) = = - . Then we say that ϕ is an isomorphism of measurable spaces. \blacklozenge 20.6 Entropy 513 Takeaways The entropy is an important characteristic of a dynamical system. For different temperatures (that is, for different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β) these can be very different values 23.4 Varadhan's Lemma and Free Energy 607 of β (red) the value of β (red) the valu 10.3. Then, for $n \in N0$, $X^n = n * E(Xi - Xi - 1)$ E(Xi - Xi - 1) E(Xi - Xi - 1) E(Xi - Xi - 1) $Ail+1 (\omega - 1)$ $Ail+1 (\omega - 1)$ Ail+parameter $\alpha > 0$; hence $P[Wn > x] = e - \alpha x$. k1 ! · · · km ! 1 In order to show (5.20), note that the event in (5.20) implies L = n and that L and (Mn, 1, . Define f (ω) = 0 for $\omega \in N$ and f (ω) = f (ω) else. Now define nk := nkk. Corollary 7.27 Let (V,) · , · *) be a linear vector space with complete semi-inner product. The following theorem confirms the conjecture mentioned above and also gives conditions under which we cannot expect that infinitely many of the events occur. U Aik). & Exercise 23.2.3 Let E = R. " = " Let X be exchangeable. We now consider σ -algebras that are generated by more than one map. In particular, Rad1/2 is called the Rademacher distribution. By definition, µ* is σ subadditive; hence we conclude by Theorem 1.36 that $\mu *$ is also σ -additive., n, be measurable spaces. Use Exercise 15.3.2 to show that $n \rightarrow \infty \mu n \rightarrow \delta 0$. If X is an L2 -martingale, then X2 is a submartingale and the corresponding increasing process is called the variance process. Since μ is additive (and thus n=1 monotone) we have by (ii) m $\mu(An) = \mu n = 1$ It follows that m An $\leq \mu(A)$ for any $m \in N$. Summing up, we have $\alpha = \lim \phi(Am) \leq \lim \phi(Am)$ -algebra = σ IA : A \in Bb (E) . 536 21 Brownian Motion Let H = L2 ([0, 1]) be the Hilbert space of square integrable (with respect to Lebesgue measure λ) functions [0, 1] \rightarrow R with inner product) f, g* = [0,1] f (x)g(x) $\lambda(dx) \sqrt{and}$ with norm f =) f, f* (compare Sect. Thus (fn)n \in N is also a Cauchy sequence in measure; hence it converges in measure by Corollary 6.15. Assume in addition that the yk are chosen such that ϕ is convex on [0, ∞). 15, we will see how the variance determines the size of the typical deviations of the arithmetic mean from the expectation. In the following, let X be a Markov chain on the countable space E with transition matrix p. It is easy to see that G is a Dynkin system. Let q1, q2, 0 Using dominated convergence, we conclude that $\lim \sup Pn([-K, K]) \le \alpha c - 1 n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \sup n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim \max n \rightarrow \infty = \alpha
- 1 \lim \max n \rightarrow \infty = \alpha - 1 \lim$ T1, T2, Determine the set of all invariant distributions. Let us construct a code $C = (c(e), e \in E)$ that is efficient in the sense that it minimizes the expected length of the code (of a random symbol) Lp (C) := pe l(e)., Vxn of E and define g := max(gx1, . (21.29) (ii) For all $\eta, \varepsilon > 0$ and $N \in N$, there is a $\delta > 0$ such that Pi $\omega : V N (\omega, \delta) > \eta \le \varepsilon$ for all $i \in N$. I. By symmetry, we have Pa $[B1-\tau > a | \tau < 1] = 12$; hence $P[B1 > a = P[B1 > a \tau < 1] P[\tau < 1] = Pa [B1-\tau > a] P[\tau < 1$ composition theorem (Theorem 1.80), for any measurable map $f: (\Omega, A) \rightarrow R$, B(R) the maps f! and "f # are also A - 2Z -measurable. Thus, * using) the tower property, we infer E[X|F] = 1 * E[X|F] + 1 = 1 * E[X|F] and "f # are also A - 2Z -measurable. Thus, * using) the tower property, we infer E[X|F] = 1 * E[X|F] + 1 = 1 * E[X|F]. efficiently. n=1 Proof " \leftarrow " Assume that (i), (ii) and (iii) hold. Choose a maximal sequence U (sorted by decreasing lengths) of disjoint intervals and show that each U \in U is in (x - 3a, x + 3a) for some (x - a, x + a) \in U . \in Z R be pairwise disjoint and let A \in Z R be pairwise d $\phi(E[X])$ be the maximal slope of a tangent of ϕ at E[X]. However, this is not very plausible. $e \in E$ $e \in E$ pe l(e) = -pe log2 (qe) $\geq H2$ (p). (ii) For $A \in A$, we denote $\{X \in A\} := P[X-1 (A) | and \theta(p) > 0$. If $g: \Omega \to [0, \infty]$ is measurable, then (by Theorem 1.4) Theorem 2.4) and $P[X \in A] := P[X-1 (A) | and \theta(p) > 0$. If $g: \Omega \to [0, \infty]$ is measurable, then (by Theorem 1.4) and $P[X \in A] := P[X-1 (A) | and \theta(p) > 0$. If $g: \Omega \to [0, \infty]$ is measurable, then (by Theorem 1.4) and $P[X \in A] := P[X-1 (A) | and \theta(p) > 0$. If $g: \Omega \to [0, \infty]$ is measurable, then (by Theorem 1.4) and $P[X \in A] := P[X-1 (A) | and \theta(p) > 0$. If $g: \Omega \to [0, \infty]$ is measurable, then (by Theorem 1.4) and $P[X \in A]$ is measurable. 4.15) $(7.4) 2\sqrt{1} = -x/2 2\pi$ g d $\nu = \text{gf d}\mu$. Then (with p(x, y) = C(x, y)/C(x)), u(1) = 1 · p(1, 2) + 0 · p(1, 0) = R(1, 0) C(1, 2) = C(1, 2) + C(1, 0) R(1, 0) + R(1, 2) R(1, 0) + R(1, 2) R(1, 0) + R(1, 2) R(1, 0) R(1, 0) + R(1, 2) R(1, 0) R(1, 0) + R(1, 2) R(1, 0) distribution π . Hence we only have to show that κ is surjective. Reflection In the situation of Theorem 19.35, come up with an example such that $n \rightarrow \infty$ E[Wi+ - Wi-] > 0 but still Xn - $\rightarrow -\infty$ holds. Using a contour argument, as for percolation (see [127]), one can show that (for $d \ge 2$) there exists a critical value $\beta c = \beta c$ (d) \in (0, ∞) such that 0 m(β) > 0 = 0, if $\beta > \beta c$, if $\beta < \beta c$, if $\beta < \beta c$, if $\beta < \beta c$, i = 1 Define probability measures Px on (Rd)N0, (B(Rd)) N0 by Px = P (S x) - 1. However, ϕ cannot be a (random) signed measure. A Definition 23.6 (Rate function) A lower semicontinuous function I : E $\rightarrow [0, \infty]$ is called a rate function. Hence A is predictable and monotone decreasing, A0 = 0, and M is a martingale. If |E| ≥ 2 , then there is no absorbing state. Repeat a random experiment with possible outcomes $e \in E$ and probabilities pe for $e \in E$ infinitely often (see Example 1.40 and Theorem 1.64)., XN) $\in A^{\sim} \epsilon$, (Xn, Xn+1, Clearly, P[A] = 12 and P[B] = 12. Hence ϕ is a topological isomorphism., in $\in I$ are pairwise distinct., $1[1/2,1)(\tau n-1(x))$ Clearly, we have #φn ([0, 1)) = #Pn., xn ∈ K such that K ⊂ U := n 1 U x j. Further, with the choice a , we can change X into a martingale. However, the characterization in terms of Laplace transforms is a bit simpler in the case of locally compact Polish spaces considered here. e 2π a Proof By the scaling property of Brownian motion (Corollary 21.12), without loss of generality, we may assume T = 1. Any subsequent person takes his or her reserved seat if it is free and otherwise picks a free seat at random. 3 $p \rightarrow \infty$ (i) If $|p| d\mu < \infty$ for some $p \in (0, \infty)$, then $f p \rightarrow f \infty$. Let pe be the probability of the symbol $e \in E$. Then $z = \lim_{n \to \infty} w_n - x = c$ by continuity of the norm (Lemma 7.23). This is possible, as F is right continuous. , μN be an orthonormal basis of left eigenvectors for the eigenvalues $\lambda 1$, . (i) (ii) (iii) (iv) B0 = 0, B has independent, stationary increments (compare Definition 9.7), Bt ~ N0,t for all t > 0, and t \rightarrow Bt is P-almost surely continuous. In order to find a necessary and sufficient condition on the growth of (wn), we need more subtle methods that appeal to the above example of the explosion of a Markov process., Xk)-measurable and Sn - Sk is σ (Xk+1, . However, this is the statement of Pólya's theorem. For A $\subset \Omega$ - , A $\in A$, we would have $\phi(A) \leq 0$ since $\alpha \geq \phi(\Omega + \cup A) = \phi($ finite contents. 3.1 Definition and Examples . A little differently from the usual convention, assume that Θ takes values in $[0, \pi]$ and $\lambda > 0$, we have $\lambda p P |X| * T \ge \lambda \le E |XT| p$. \bullet 14.1 Product Spaces 305 Definition 14.6 Let $I = \emptyset$ be an arbitrary index set, let (E, E) be a measurable space, let $(\Omega, A) = (E I, E \otimes I)$ and let $Xt : \Omega \to E$ be the coordinate map for every $t \in I$. We will see that this is indeed possible. (13.2) 276 13 Convergence of Measures on (E, E), Mf(E) := Radon measures on (E, E), Mf(E) := finite measures on (E, E), $M1(E) := \mu \in Mf(E) := \mu \in Mf(E)$. 1, $M \le 1$ (E) := $\mu \in Mf(E)$: $\mu(E) \le 1$. Define $Xn\lambda = \lambda$, Xn *, $Sn\lambda = \lambda$, Sn* * and $S\infty \sim N0, C$. (i) Show by a direct computation using only the definition of stability that $|\phi(t) - 1| \le C|t|\alpha$ for t close to 0 (for some $C < \infty$). Let $A \in A$ with $\mu(A) = 0.3$ Then $Z = \{A, Ac\} \in Z$ and fZ = 1 Ac $\nu(Ac)/\mu(Ac)$. As E is locally compact, there is a compact set L with $K \subset A$ with $\mu(A) = 0.3$ Then $Z = \{A, Ac\} \in Z$ and fZ = 1 Ac $\nu(Ac)/\mu(Ac)$. As E is locally compact, there is a compact set L with $K \subset A$ with $\mu(A) = 0.3$ Then $Z = \{A, Ac\} \in Z$ and fZ = 1 Ac $\nu(Ac)/\mu(Ac)$. $L^{\circ} \subset L \subset G$. Proof Let ϕ be such a coupling. The distribution
of F0 and) · , · *0. is a martingale. i \in I (iii) Let μ be a σ -finite measure on A, and assume every Ei is also a π -system. 19.2 Parallel connection of six resistors., $jn \in \{1, . Note that \phi(t) = \psi(t) \text{ for } |t| \le \pi/2 \text{ and } \phi(t) = 0 \text{ for } |t| > \pi/2; \text{ hence } \phi \ge \phi \cdot \psi. \text{ Now apply (16.2) with } zn = |\phi n(t)|^2$. (iii) Determine E[ei t1 X1 + i t2 X2] and E[ei t1 X1 + i t2 X2] for t1 = 12 and t2 = 2. For every $\omega \in \Omega$, we say that the map I \rightarrow E, t \rightarrow Xt (ω) is a path of X. Hence also (Xn (k), k = 1, . 0 = v(An) - v(An) = An By Theorem 4.8(i), f2 1An = f1 1An μ -a.e. As f1 > f2 on An, we infer $\mu(An) = 0$ and $\mu(\{f1 > f2\}) = \mu$ An = 0. Further, we write f (x) = f(x1, ... 367 16.2 Stable Distribution) If X and Y are independent exp θ distributed random variables, then it is easy to check that $X - Y \sim \exp 2\theta$. A map $\phi: E \rightarrow E$ is called Hölder-continuous of order γ (briefly, Hölder- γ -continuous) at the point $r \in E$ if there exist $\epsilon > 0$ and $C < \infty$ such that, for any $s \in E$ with $d(s, r) < \epsilon$, we have $d(\phi(r), \phi(s)) \leq C d(r, s)\gamma$. 286 13 Convergence of Measures D Note that (13.7) implies $F(\infty) \leq \lim \inf n f \rightarrow \infty$ Fn (∞) Then *) Sn-1 + M. i=1 (ii) There is a modification of W such that $t \rightarrow Wt$ is almost surely continuous (see Remark 21.7). By Theorem 7.33, we get that $\nu = \nu_a$ has a density with respect to μ . We say that ($\mu \epsilon$) $\epsilon > 0$ satisfies a large deviations principle (LDP) with rate function I if (LDP 1) lim inf ϵ log($\mu \epsilon$ (U)) $\geq -$ inf I (U) for every open U $\subset E$, (LDP 2) lim sup ε log($\mu\varepsilon$ (C)) $\leq -\inf I$ (C) for every closed C $\subset E$. (Here P could be an arbitrary measure on (Ω , A).) * (ii) If (An) $n\in N$ is independent and ∞ n=1 P[A] = ∞ , then P[A] = 1. Each of the x individuals has an exponentially distributed lifetime with parameter 1., 2n }, n $\geq n0$. Theorem 8.36 Let E be a Polish space with Borel σ -algebra E. The numbers r1, r2, On the other hand, ∞) *) * E 1A Yn E[X |F] = E 1A 1{Yn = k 2-n} k 2-n E[X |F] = E 1A 1{Yn = k 2-n} k 2-n X k=1) * n \rightarrow \infty = E 1A Yn X - \rightarrow E[1A Y X]. Then there exists an $F \in V$ and a subsequence (Fnk) k $\in N$ with $k \rightarrow \infty$ Fnk (x) $\rightarrow F$ (x) at all points of continuity of F. 1.1 Classes of Sets In the following, let $\Omega = \emptyset$ be a nonempty set and let $A \subset 2\Omega$ (set of all subsets of Ω) be a class of subsets of Ω . Then $X\sigma \ge E[X\tau \ F\sigma]$, and, in particular, $E[X\sigma \] \ge E[X\tau \ F\sigma]$, and, in particular, $E[X\sigma \] \ge E[X\tau \ F\sigma]$, and in particular, $E[X\sigma \] \ge E[X\tau \ F\sigma]$. First consider a i.i.d. sequence of random variables with values in a finite set. For any $x \in \mathbb{R}$, we have closely related to Wright's model, we can argue that the limit $X \propto can$ take only the stable values 0 and 1. Hence Evidently, $Z \circ \tau = Z$; hence $F \in I$. We first investigate convergence with respect to $\cdot p$. The powers of M are easy to compute: $M\psi = 0.1$, -1.2 $M\psi 2 = -1.2$, -2.3 $M\psi 3 = -2.3$, -3.4 and inductively $M\psi n = -(n-1) - n.n.n+1$. e-cn/2 k=-n Therefore, $-RW = \infty$ 1 $k-1 n-0 -1 \le n=1 k=-n 1$ $k-1 n-0 -1 \le n=1 k=-n + 0$ (ω) with n k > ecn/2 for all $n \ge n+0$. In the first section, we study which probability measures on R are infinitely divisible and give an exhaustive description of this class of distributions by means of the Lévy-Khinchin formula. This is often called the mean field assumption. are i.i.d. real random variables, then (PSn /n) $\in N$ satisfies an LDP with rate function $\Lambda *$. • 19.4 Recurrence and Transience 0 479 1 R(0, 1) = 1/3 2 R(1, 2) = 2/9 3 R(2, 3) = 4/27 Reff (0 $\leftrightarrow 3$) = 19/27 Fig. Reflection In the proof of Theorem 5.10, where did we use the independence of T? Hence the measure PX is characterized by its values on I. Then, by symmetry, also $f \ge 0$ and hence $f \equiv 0$. If E has a finite diameter diam(E), then dW (P, Q) \le (diam(E) + 1)dP (P, Q) for all P, $Q \in M1$ (E). n! Thus, for $|h| < 1/(3\alpha)$, $n * *1/n > |h| \cdot e/n \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim \sup (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E |X|n n \to \infty = \lim (2\pi n) - 1/2 E$ n -1/2 (e/3)n = 0. Hint: Let $\pi = 0$ be an invariant measure for X and abbreviate $P\pi = \pi({x})Px x \in E$ (note that, in general, this need not be a finite measure). and thus $h(P, \tau) = 0$. Since f $nk \to \infty$, we can choose a subsequence (fnkl) $l \in N$ and the following, we assume that there exists an $\alpha \in I$ (0, 2] such that U (x) x $\alpha - 2$ is slowly varying at ∞ . By Lemma 4.6(iii), we infer $(f + g) + d\mu + f - d\mu + Hence$ $(f + g) + d\mu + f - d\mu + Hence$ $(f + g) + d\mu + f - d\mu + g + d\mu$. Theorem 1.15 (Intersection of classes of sets) Let I be an arbitrary index set, and assume that Ai is a σ -algebra for every $i \in I$. The probabilistic meaning of this fact is that as a continuous function $\log(\phi(t))$ is uniquely defined and thus there exists only one continuous function $\phi 1/n = \exp(\log(\phi)/n)$. 442 18 Convergence of Markov Chains Y is independent of By Step 1, there exists a successful coupling (X, X' and Z1, Z2, . The family (Ei)i \in I is called independent if, for any finite subset J \subset l and any choice of $Ei \in Ei$, $i \in I$, we have $P + j \in I$, Ej = P[Ei]. (i) F is the distribution function of a 12 -stable distribution. Here $\varphi(t) = D1 D i = 1 cos(ti)$. For any c > 0, we have $|f - g|p = |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1\{|f - g| > c\} \le cp + cp - p |f - g|p 1|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p 1|p |f - g| > c\} \le cp + cp - p |f - g|p |f - g$ sub-σ -algebra and let (Ai)i EI be an arbitrary family of sub-σ -algebras of F.
Takeaways For a Markov process, the conditional distribution at time t given the full history until some time s < t is a function of the state at time t given the full history and can be described by a stochastic kernel κs,t. Further, let D1, . What is the probability that the first position changes for a Markov process, the conditional distribution at time t given the full history until some time t given the full history and can be described by a stochastic kernel κs,t. infinitely often? It is natural to use the entropy in order to quantify also the randomness of a dynamical system. 413.2 Weak and Vague Convergence 289 Exercise 13.2.8 Show that (13.4) defines a metric on M1 (E) and that this metric induces the topology of weak convergence. It is easy to see that $\alpha := \inf\{h(x) : |x| \ge 1\} = 1 - \sin(1) > 0$. Clearly, it is a crucial requirement for any strategy that the decision for the next stake depend only on the information available at that time and not depend on the future results of the qamble. k, N=1 Hence the claim follows., p-1, $n \rightarrow \infty \log(d+1) - \log(d) 1 \# i \le n$: ai = $d \rightarrow \cdot$. Second strategy that the decision for the next stake depend on the future results of the qamble. k, N=1 Hence the claim follows., p-1, $n \rightarrow \infty \log(d+1) - \log(d) 1 \# i \le n$: ai = $d \rightarrow \cdot$. (µ). Then $\lim n \to \infty$ fn dµ = f dµ, where both sides can equal + ∞ . For $\lambda \ge 0$, applying (21.45) twice yields $\kappa t(x, dy) \kappa y(y, dz) = -\lambda z = \lambda y \kappa t(x, dy) \kappa t(x, dy) \kappa y(y, dz) = -\lambda z = \lambda y \kappa t(x, dy) \kappa t$ $\log Pn((-\infty, x)) \ge -\inf I(y)$ for x < 0. What is the voltage u(x) at $x \in E \setminus A$? Show that the statement of the source coding theorem holds for b-adic prefix codes with H2 (p) replaced by Hb (p). n=0 Takeaways A state of a Markov chain is called recurrent if the chain returns to it almost surely. Morse alphabet. \circ F1 (x). Then u(x) = Ex [u(X τA)]. Reflection Can we relax the condition in (viii) that the Xn be dominated to uniform integrability of (Xn)n \in A either $\mu(A) = 0$ or $\delta(\epsilon) > 0$ as in (ii) and C as C in (i). Assume there exists an a > 0 such that for any $A \in A$ either $\mu(A) = 0$ or $\delta(\epsilon) > 0$ as in (ii) and C as C in (i). $\mu(A) \ge a. 2$ Lemma 15.47 If (i) of Theorem 15.44 holds, then kn)* lim log $\phi_n(t) - E$ eit Xn, l - 1 = 0. * 7.4 Lebesgue's Decomposition Theorem In this section, we employ the properties of Hilbert spaces that we derived in the last section in order to decompose a measure into a singular part and a part that is absolutely continuous, both with respect to a second given measure. \bigstar Exercise 17.3.3 Show that, almost surely, infinitely many balls of each color are ∞ 1 drawn if = ∞ . (ii) For almost all $\omega \in \Omega$, the map $x \to f(\omega, x)$ is continuous at the point x0. For simplicity, assume that $\Lambda = \{0, ., i=1, ..., n \}$ 11.20 (Kesten-Stigum [95]) Let m > 1. Formally, the set of edges is a subset of the set of subsets of Zd with $x - y^2 = 1$. Then P[Nn,t +1 ≥ 1] = E[hn,l (Nn,t)] and hn,l (k) is monotone increasing both in k and in n. l=1 Hence Ym is σ (Xm,l, $l \in N$)-measurable and thus (Ym)m $\in N$ is independent. Compute ψ (1) and the extinction probability. For any $\omega \in \Omega$, let $\kappa(\omega, \cdot)$ be the probability measure on (Ω, A) with distribution function $F^{-}(\cdot, \omega)$. (Hint: Use Exercise 21.2.3 and the optional sampling theorem.) (ii) τb has a 12 -stable distribution with Lévy measure $\sqrt{\nu(dx)} = b/(2\pi) \times -3/21 \{x>0\}$ dx. Let $F0 = \{\emptyset, \Omega\}$ and $Fn = \sigma(Y1, .)$ Define I := := and $N = 0 Q \cap I$. $(Q \cap I) \cup \max I$, $r \in I Nr .$, XT). Lemma 15.50 If (i) of Theorem 15.44 holds, then $n \rightarrow \infty$ ft $d\nu n \rightarrow -t2$. Thus, the theorem will become clear through the following observation. Hence $\mu(B) = \mu(A) + ni = 1$ $\mu(Ci) \ge \mu(A)$ and thus μ is monotone. We say that (Xn, I) = Xn, I = 1. Definition 18.1 (i) For x, $y \in E$, define N(x, y) := n \in N0 : pn (x, y) > 0. Define ST := T Xi. Then I (x1) is the total flow into the network and -I(x0) is the total flow into the network and -I(x0) is the total flow into the network and -I(x0) is the total flow into the network and -I(x0) is the total flow into the network. this case, we denote $Z \circ X - 1 := \phi$ (even if the inverse map X - 1 itself does not exist). Proof Let k1, Let Hn be the number of euros to bet in the $P\Xi \infty = w$ -lim $P\xi n$ (X) for a suitable subsequence (nl) $\in N$. all $A \in A$ with $\mu(A) < \delta/(2K)$, we obtain $h \, d\mu \le K\mu(A) + h \, d\mu < \delta$; $\{h \ge K\}$ A hence $3 \land |f| \, d\mu \le \varepsilon$ for all $f \in F$. Then f is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ A hence $3 \land |f| \, d\mu \le \varepsilon$ for all $f \in F$. Then f is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ A hence $3 \land |f| \, d\mu \le \varepsilon$ for all $f \in F$. Then f is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ A hence $3 \land |f| \, d\mu \le \varepsilon$ for all $f \in F$. Then f is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ A hence $3 \land |f| \, d\mu \le \varepsilon$ for all $f \in F$. Then f is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ A hence $3 \land |f| \, d\mu \le \varepsilon$ for all $f \in F$. Then f is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probability measure. Finally, for $k \in \{1, ..., K\}$ is the characteristic function of a probabilit $(\omega) - Xt(\omega 0) | \land 1$ is A-measurable. This is trivial. Theorem 12.26 (de Finetti representation theorem) The family $X = (Xn)n \in N$ is exchangeable if and only if there is a σ -algebra $A \subset F$ and an A-measurable random variable $\Xi \infty : \Omega \to M1$ (E) with the property that given $\Xi \infty : (Xn)n \in N$ is exchangeable if and only if there is a σ -algebra $A \subset F$ and an A-measurable random variable $\Xi \infty : \Omega \to M1$ (E) with the property that given $\Xi \infty : (Xn)n \in N$ is i.i.d. with $L[X1 | \Xi \infty] = \Xi \infty : \rho - 1$. Then $\psi(x, y) = x 1/p + y 1/p$ is Fig. Example 20.31 Assume that P is a product measure with marginals π on E. Then (since E[Xn+1 - Xn Fn] = 0) E[(H · X)n + Hn+1 E[Xn+1 - Xn Fn] = (H · X)n + Hn+1 E[Xn+1 - Xn Fn] = (H · X)n + Hn+1 E[Xn+1 - Xn Fn] = (H · X)n +
Hn+1 E[Xn+1 - Xn Fn] = (H · X)n + Hn+1 E[Xn+1 - Xn dependent, but stationary, random variables. t almost surely and hence X is a modification of X. Why is it not enough to assume in the theorem that (Xn)n \in N be uncorrelated (instead of pairwise independent)? & Exercise 18.4.2 Show (18.17). 18.3 Markov Chain Monte Carlo Method ... Hint: Show that the difference of two independent recurrent random walks is a recurrent random walk. By the one-dimensional central limit theorem, $n \rightarrow \infty$ we have PSn $\lambda \rightarrow N0$, $\lambda, C\lambda^* = P$, $\Delta \infty^*$. Definition 1.103 (Distributions) Let X be a random variable., Dn. Here we have the connection to Shannon's theorem. In particular, un (1) > 0 for sufficiently large n. We can compute the effective resistances in parallel and serial connections. $n \rightarrow \infty$ (iv) $\mu = v$ -lim μn and { μn , $n \in N$ } is tight., XT -1, ±1) and VT ± = gT (X1, . are identically distributed. p) *) * (ii) For any p > 1, we have thus shown the following theorem of Pólya [134]. In this case, it is enough to compute at each step F (0) and F (1) since F is constant if the values coincide. Definition 2.3 (Independence of events) Let I be an arbitrary index set and let (Ai)i \in I be an arbitrary family of events. 1 n n k=0 Xk diverges 16.2 Stable Distributions 389 Hint: Compute the density of F , and show that the Laplace transform is given by $\sqrt{\lambda} \rightarrow e - 2\lambda$. If in addition μ is σ -subadditive, then $\mu *$ (A) = $\mu(A)$ for all $A \in A$. \downarrow lim $n \to \infty$ $n \to \infty$ integrable function on Exercise 4.2.3 Let $f \in L1$ ([0, ∞), λ) be a Lebesgue [0, ∞). For any weak 366 15 Characteristic Functions and the Central Limit Theorem limit point O for (PXn) $n \in \mathbb{N}$ and for any $\lambda \in \mathbb{R}^d$, we have) $\lambda^* O(dx) ei \lambda x^* = E eiX$. The general case, can be inferred by the usual approximation arguments (see Theorem 1.96((i))). As a shorthand, we write sup $L(\phi)$ for the map $x \rightarrow \sup\{f(x) : f \in L(\phi)\}$. As F is tight, for every $\varepsilon > 0$, there is a $K < \infty$ with Fn $(x) - Fn(-\infty) < \varepsilon$ for all $n \in N$ and x < -K. 3 3 3 (a) 3 $(f + g) d\mu$ 3 = f $d\mu + g d\mu$. X is called the multi-period binomial model or the Cox-Ross-Rubinstein model (without interest returns). Now $\sigma(\{Fn, n \in N\}) = 2\Omega$, and 22 1 Basic Measure Theory hence $E = \{Fn, n \in N\}$ is a π -system that generates 2 Ω and such that $\mu(Fn) < \infty$ for all $n \in N$. " \leftarrow " This is trivial. Show that the map $\Phi : L1(F\infty) \to M, X\infty \to (E[X\infty|Fn])n \in N$ is an isomorphism of vector spaces. For $\varepsilon > 0$, let $\mu \{|fm - fn| > \varepsilon\} \le \varepsilon - 1$ fm $- fn 1 \to 0$ for m, $n \to \infty$. Assume that the following conditions are satisfied. The Lebesgue integral does not do that. Compare Example 12.3(iii). Taking the formal derivative $\infty \sqrt{d X^2 t} := Xt = \xi 0 + 2 \xi n \cos(n\pi t) dt n = 1$ we get independent identically distributed Fourier coefficients for all frequencies. In particular, the assumption that the factors are of Lebesgue - Stieltjes type can be dropped. + Xn)2k-1 \leq d2k-1 nk-1 and)* (2k)!)*k E (X1 + . Proof The first equation holds by definition. The probabilities of the observations will be described in terms of the distribution of the corresponding random variable, which is the image measure of P under X. Now lim inf n $\rightarrow \infty$ 1 log Pn (x - ε , ∞) \geq -I (x) > -I (x + ε) n 1 log Pn [x + ε , ∞). (ii) If σ , $\tau \ge 0$, then $\sigma + \tau$ is also a stopping time. Evidently, $\mu * (\emptyset) = 0$ and $\mu *$ is monotone. Then $1 + 1 - \infty Xk = f \circ \tau k \rightarrow E[X0]$ P-a.s. k=0 Proof If τ is ergodic, then E[X0] = E[X0] and the supplement is a consequence of the first statement. Assume for every T > 0, there are numbers α , β , C > 0 such that *) E |Xt - Xs | $\alpha \leq C|t - s|1+\beta$ for all s, $t \in [0, T]$. Then X is called *) *) (v) stationary if L (Xs+t) t $\in I = L$ (Xt) $t \in I$ and (vi) a process with stationary if L (Xs+t) t $\in I = L$ (Xt) $t \in I$ and (vi) a process with stationary if L (Xs+t) t $\in I = L$ (Xt) $t \in I$ and (vi) a process with stationary increments if X is real-valued and) *) * L Xs+t + r - Xt + r = L Xs+r - Xr (If $0 \in I$, then it is enough to consider r = 0.) for all r, s, t \in I. \blacklozenge Theorem 15.6 (Laplace transform) A finite measure μ on $[0, \infty)$ is characterized by its Laplace transforms LY1, Then E[X1] = $-i \phi$ (0) < ∞ . (14.18) (14.19) To this end, define stochastic kernels κt (x, dy) := $\delta x * N0, t$ (dy) for $t \in [0, \infty)$ where N0,0 = $\delta 0 \cdot \infty$ (A eit) it = eA(e -1). Exercise 3.3.1 Assume that we have a branching process Z = (Zn)n \in N0 with Z0 = 1 whose offspring distribution is given by p0 = 1/3 and p2 = 2/3. Reflection Check that each of the families E1, $\bullet \bullet$ Owing to the last theorem, it is natural to define the convolution of two probability measures on Rn (or more generally on an Abelian group) as the distribution of the sum of two independent random variables (Xn+)n eN and (Xn-)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN and (Xn-)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN and (Xn-)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise independent families of square integrable random variables (Xn+)n eN again form pairwise integrable random variables (Xn+)n eN again form pairwise integrable random variables (Xn+)n eN again form pairwise integrable random variables (Xn+)n eN again form pairw other end of the scale, the strongest notion is "i.i.d.". A locally compact, separable metric space is manifestly σ -compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology
consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of relatively compact and there even exists a countable basis U of the topology consisting of the topology consist X be an N0 -valued random variable. Show that Y has distribution Q., $\omega n \in \{0, 1\}$, $n \in N$. $\in A$ with AN $\uparrow \Omega$ and $\mu(AN) < \infty$ for all $t \ge 0$. P (iii) Let $n = T\epsilon$. (23.21) 606 23 Large Deviations Proof This is left as an exercise. Proof By Theorem 14.12(iii), the finite-dimensional distributions determine P uniquely., ωn is compact. (ii) Let (Yn)n $\in Z$ be i.i.d. real random variables and let c1, Remark 7.48 As F is continuous, for any $\delta > 0$, there exists an $\varepsilon > 0$ such that $|F(f)| < \delta$ for all $f \in V$ with $f < \varepsilon$. Then: (i) $F := \{r \in [0, 1] : \psi(r) = r\} = \{q, 1\}$. values of the Xi by size M = X(1) > D X (2) > . , Sn $\{-Mn \circ \tau \}$ (4 + $x \circ (1 + x) \alpha$ is holomorphic up to possibly a singularity at x = -1. We call (Zd, Ep) a percolation model (more precisely, a model (more precisely, a model) * = E[Mn] - E[Mn]for bond percolation, in contrast to site percolation, where vertices can be open or closed), 18.1 Periodicity of Markov Chains We study the conditions under which a positive recurrent Markov chains We study the conditions under which a positive recurrent Markov chains $\mu \in M1$ (E), converges in distribution to an invariant distribution π ; that is, $n \rightarrow \infty$ µpn $- \rightarrow \pi$. Now assume inf f (K) < ∞ . \blacklozenge However, this is the CFP of br, $p \nu$ r, p Not every infinitely divisible distribution is of the type CPoi ν , however we have the following theorem., Mn, $m = km \cdot P[L = n] n n! km - \alpha \alpha p 1 k1 \cdots pm e k1 ! \cdots km ! n!$ m k i $\lambda e - \lambda i$ i. By what we have shown already, there exists a compact set $C \subset A \cap Kn$ with $\lambda((A \cap Kn) \setminus C) < 1$; hence $\lambda(C) > L$. Exercise 11.2.1 For p = 1, the statement of Theorem 11.10 may fail. By Theorem 2.26, the two random variables are independent, and *) *) *) E (Sk + c) 1Ak E Sn - Sk = 0. n \rightarrow \infty By assumption, $|fn| \le h$ and $fn \rightarrow f(\cdot, x0)$ almost everywhere. Let E := q - 1 ({1}). That is, e - t Xtconverges in distribution to a random variable W with W ~ exp1. The Markov process X on N0 with this semigroup is called a Poisson process with (jump) rate θ . (i) X is called a random variable with values in (Ω , A). Proof A = 2 Ω is a σ -algebra with E \subset A. Let N := max n \in N0 : X1 + . Hence there exists an n0 = 1 we conclude that n1 n-1 A i=0 i pn pnc – ε n0 (ω) such that n-1 i=0 1Ai ≥ 2 for all n \geq n0. By Corollary 9.34, Y is a submartingale, and (by Theorem 9.39) so are H · Y and (1 – H)·Y. By Remark 1.17, this implies σ (E1) $\subset \sigma$ (E4). 6.3 Exchanging Integral and Differentiation 161 By assumption (ii), we have n $\rightarrow \infty$ gn – \rightarrow f (·, x0) μ -almost everywhere. We define F := σ (Bi , i \in I). Assuming rs = 0 for all $s \le (n - 1)t$, we conclude $nt sup rs \infty \le 2\lambda ru \infty du \le 2\lambda t sup rs \infty = 0$, $s \le nt s \le nt$ (Al) = limN $\rightarrow \infty EN$ (Al) exists in a suitable sense. n=1 We have to show $A \in M(\mu *)$; that is, $\mu * (A \cap E) \le \mu * (E)$ Let Bn = n for any $E \in 2\Omega$. In Definition 4.16, for measurable $f: \Omega \to R$, we defined $f p := 1/p | f | d\mu p$ for $p \in [1, \infty)$ and $f \infty := \inf K \ge 0$. Step 2. Apply a voltage of 1 at the origin and 0 at the endpoints of the paths at the nth stage. (Hint: Assume the contrary and show that the corresponding random variable would have variance zero.) A α Exercise 15.4.4 Let X1, X2, (16.32) Theorem 16.28 (i) If PX is in the domain of attraction of some distribution, then there exists an $\alpha \in (0, 2]$ such that (16.32) holds. are i.i.d. and $\sim \text{Rad}/2$; that is, P[Ri = 1] = 1 - P[Ri = -1] = 1. We show P[A] = 0. Is this true also for P[$\cdot |X = x$]? By Lemma 1.51, μ is a measure and μ is σ -finite. Let An = En $\cap \{f1 > f2\}$ for $n \in N$. Without loss of generality, we may assume that X is the canonical process on the probability space (Ω , A, P) = E N0, B(E) × N0, P = E N0, P convergence in measure: For $\varepsilon > 0$, define Dn (ε) = d(f, fm) > ε for some m \ge n. (18.17) Apart from the double zeros σ cos $\pi k/N$, k = 1, . -n be Exercise 7.4.1 For every x \in (0, 1], let x = (0, x1 x2 x3. Definition 1.29 Let A be a semiring. However, since X is a supermartingale, for every s > t, we have Xt \ge E[Xs |Ft] Q+ s \downarrow t, s>t $- \rightarrow t$ [Ft] = X t E[X in L1 . k=1 For c = Var[Sn]/t ≥ 0 , we obtain P[A] $\leq Var[Sn] + c2 = 2$. Show that $\lambda = v$ -lim µn but that (µn) $n \in N$ does not converge weakly. + Tn-1 $\leq t$ and s $\leq t$. k = 1,...,n 5.5 The Poisson Process 139 5.5 The Poisson Process We develop a model for the number of clicks of a Geiger counter in the (time) interval I = (a, b]., d, the map yi \rightarrow f (x1, Definition 15.21 Let (E, d) be a metric space. < yN of F such that F (y0) < ϵ , F (yN) > F (∞) - ϵ and yi - yi-1 < ϵ for all i. If there is no infinite open component, then the water may wet only a thin layer at the surface. Hence we have constructed a (random) subgraph (Zd, Ep) of (Zd, E). 440 18 > 0, let U_E (t) := {s \in I : |s - t| < s}. What goes wrong if \cap -stability is missing? Indeed, let fn \rightarrow f and meas fn \rightarrow g. (23.15) Proof We consider the set of possible values for the n-tuple (X1, . * Example 5.21 (Monte Carlo integration) Let f : [0, 1] \rightarrow R be a function and 31 assume we want to determine the value of its integral I := 0 f (x) dx numerically. Theorem measure) There exists a uniquely determined measure 1.55 (Lebesgue λn on Rn, B(Rn) with the property that λn ((a, b]) = n (bi - ai) for all a, b \in Rn with a < b. Other important convergence theorems for integrals follow in Chaps. n If m, n \in N, then write mn if m is a divisor of n; that is, if m \in N., $\alpha m \in (0, \infty)$, and for mutually disjoint sets A1, Proof This is obvious. In order to show that DE is a λ -system, we check the properties of Definition 1.10: (i) Clearly, $\Omega \in DE$. By Theorem 13.16, it is enough to show that $n \rightarrow \infty$ f dµn $- \rightarrow$ f dµ. Then the bivariate Markov chain Z := ((Xⁿ, Yⁿ))n \in NO has the transition matrix p defined by p (x1, y1), (x2, y2) = p(x1, x2) p(y1, y2). Finally, there are more exercises and some new illustrations. A prominent role will be played by the complex exponential function exp : $C \rightarrow C$, which can be defined either by Euler's formula exp(z) = $\infty n=0 z/n!$. $\tilde{B} \subset \{1, ..., N=0\}$ Network Reduction Example 19.32 Consider a random walk on the graph in Fig. The statement can be derived more simply than by direct computation if *n (see Example 3.4(ii)). For continuous time, however, an infinitely divisible real random variable X is not simply the difference of two infinitely divisible nonnegative random variables, as the normal distribution shows. & Chapter 3 Generating Functions It is a fundamental principle of mathematics to map a class of objects that are of interest into a class of objects that are of interest into a class of objects that are of interest into a class of objects where computations are easier. It should • be random and independent for disjoint intervals, • be homogeneous in time in the sense that the number of clicks in I = (a, b] has the same distribution as the number of clicks in c + I = (a + c, b + c], • have finite expectation, and • have no double points: At any point of time, the counter makes at most one click. Let $f: Z \rightarrow R$ be an arbitrary map. Then z = 0 and F(z) = F(v) - F(v) = 0., n/let (Ωi , A/ i, Pi) be a probability space. We can interpret X + 1 as the waiting time for the first success in a series of independent random experiments, any of which yields a success with probability p) or removed., 0) and x N = (0, . Proof (i), (ii) and (iii) These are evident. However, we show that, for open A this can be done at least asymptotically. For example, in the preceding theorem, claim (i) holds with the words "submartingale" and "supermartingale" and "supermartingale" interchanged, claim (iv) holds for submartingale" and "supermartingale" interchanged, claim (iv) holds for submartingale" and "supermartingale" interchanged, claim (iv) holds for submartingale if the minimum is replaced by a maximum, and so
on. Exercise 20.5.1 Show that "strongly mixing" implies "weakly mixing" implies "weakly mixing" implies "submartingale" interchanged, claim (iv) holds for submartingale if the minimum is replaced by a maximum, and so on.

'ergodic''. If) · , ·* is an inner product, then (V,) · , ·*) is called a (real) Hilbert space if the norm defined by x :=)x, x*1/2 is complete; that is, if (V, ·) is a Banach space. Definition 2.32 (Convolution of measures) Let μ and PY = ν . 22, we will see that for independent identically distributed, square integrable, centered random variables X1, X2, Although it is sometimes convenient to allow also time-inhomogeneous Markov processes. Let $v \in V \setminus W$ and let v = y + z for $y \in W$ and $z \in W \perp$ be the orthogonal decomposition of v. (i) If $E[|X|n] < \infty$, then ϕ is n-times continuously differentiable with derivatives *) $\phi(k)(t) = E$ (iX)k eit X for k = 0, be measurable maps $\Omega \rightarrow [0, \infty]$. (ii) Conclude that Prohorov's theorem holds for E = Rd. Thus X is recognized as a Markov process. are not independent but only pairwise independent, then the rate of convergence deteriorates, although not drastically. C 21.5 Construction via L2 -Approximation 537 Hence Xtn n \in N is a Cauchy sequence in L2 (P) is complete, see Theorem 7.3) has an L2 -limit Xt. First generate the transition matrix p of the Markov chain. Let A, B $\subset \Omega = E$ N0 be measurable. y $\downarrow x$ (iii) For x \in I \circ , we have D - $\phi(x) \le D + \phi(x)$ and $\phi(x) + (y - x)t \le \phi(y)$ for any $y \in I \iff t \in [D - \phi(x), D + \phi(x)]$. Ik (Note that it is not a priori clear, that the logarithm in the two equations above is well-defined. Then $(Y_n = -e) = (p + \phi(x) + \phi(x)]$. Ik (Note that it is not a priori clear, that the logarithm in the two equations above is well-defined. Then $(Y_n = -e) = (p + \phi(x) + \phi(x)]$. (compare Definition 5.25). (ii) Gamma distribution: Γθ,r * Γθ,s = Γθ,r+s for all θ, r, s > 0. Without proof, we quote the following strengthening of Corollary 16.8 that relies on a finer analysis using the arguments from the proof of Theorem 16.12 Let (φn,l; l = 1, . This is the reason for phase transitions at critical temperatures (e.g., melting ice). Let $n \in N$ and let $(\Omega 2, A 2) = \{0, 1\}n$, $(2\{0,1\}) \otimes n$ be the space of n-fold coin tossing. be i.i.d. random variables with values in the finite set Σ and with distribution μ . Lemma 1.49 $A \in M(\mu *)$ if and only if $\mu * (A \cap E) + \mu * (A \cap E) + \mu$ the face only shows a number one to five. The nth convolution roots are thus unique if the distribution is infinitely divisible., be i.i.d. random variables with distribution $\mu\alpha$. Klenke, Probability Theory, Universitext, 229 230 10 Optional Sampling Theorems Theorem 10.1 (Doob decomposition) Let X = (Xn)n \in N0 be an adapted integrable process. While the Lebesgue measure is ubiguitous in analysis, the infinite product measure plays an important role in probability theory for modelling infinitely many independent events. Definition 1.20 (Topology) Let $\Omega = \emptyset$ be an arbitrary set. \blacklozenge Reflection Let $X = (Xn) n \in Z$ be a reversible Markov chain (with transition matrix p) with respect to π and assume that $PX0 = \pi$. Theorem 7.21) · , · * is an inner product on L2 (µ) and a semi-inner product on L2 (µ). That is, fX , fY : Rn $\rightarrow [0, \infty]$ are measurable and integrable with respect to n-dimensional Lebesgue measurable with respect to n-dimensional Lebesgue measurable and integrable with respect to n-dimensional Lebesgue measurable and integrable with respect to n-dimensional Lebesgue measurable and integrable with respect to n-dimensional Lebesgue measurable with respect to n-dimensional Lebesgue measurable and integrable with respect to n-dimen to A - Aj. Hence we now consider the case where only two values are possible. Together with stochastic convergence, uniform integrability is equivalent to L1 - convergence, uniform integrability is equivalent to L1 - convergence, uniform integrability is equivalent to L1 - convergence. $\mu F = \infty n = 1 \alpha n \delta xn$. Define the composition of $\kappa 1$ and $\kappa 2$ by $\kappa 1 \cdot \kappa 2 : \Omega 0 \times A2 \rightarrow [0, \infty)$, $(\omega 0, A2) \rightarrow \kappa 1 (\omega 0, d\omega 1) \kappa 2 (\omega 1, A2)$. Definition 9.1 (Stochastic process) Let $I \subset R$. We conclude that P[F] = 0.1.2 Set Functions 13 Then A is an algebra. "(iii) \Rightarrow (i)" This is trivial. Then $|fn(s) - f(s)| \leq |fn(s) - f(s)| + |f(ti) - f(ti)| + |f(ti) - f(s)| \leq 3\epsilon$. Hence $(kt) t \in I$ is a Markov semigroup. Remark 1.22 In many cases, we are interested in B(Rn), where Rn is equipped with the Euclidean distance n d(x, y) = x - y2 = $(xi - yi)^2$. Since ψ is monotone increasing, it follows that $r = \psi(r) \ge \psi(q0) = q1$. Then, for any t > 0, $y = x - y2 = (xi - yi)^2$. (ii). If B(n) denotes the set of points up to the nth generation, then n-1 n-1 Reff 2k 3-k. Proof By Lemma 15.37 and Lévy's continuity theorem (Theorem 15.24), PSn* $n \rightarrow \infty$ a converges to the distribution with characteristic function $\phi(t) = e - t/2$. In the Borel-Cantelli lemma we have encountered a special case. ., and let X := $\alpha + \infty Xk$. How can we quantify the information inherent in a message X1 (ω), The problem consists in finding a candidate for a weak limit point. X and Y are called uncorrelated if Cov[X, Y] = 0 and correlated otherwise., ωn]: $\omega 1$, Then X is not stationary. Denote by $\tau := \inf\{n \in N0 : Xn = Yn\}$ the time of coalescence. Since E[Zn-1] = mn-1 (Lemma 11.18), by the Blackwell- Girshick formula (Theorem 5.10), $Var[Wn] = m-2n \sigma 2 E[Zn-1] + m2 Var[Zn-1] = \sigma 2 m-(n+1) + Var[Wn-1]$. Rotations are not. Then, for $n \ge m$, E[Sn Fm] = E[X1 Fm] + .1.3 The Measure Extension Theorem 27 Further choose $a \in (a, b)$ such that $\mu((a \varepsilon, b]) \geq \mu((a, b]) - 2\varepsilon . n n \rightarrow \infty n By the strong law of large numbers, we have limn \rightarrow \infty n 1 Bn = 0 a.s. Using a generalization of the reflection principle (Theorem 17.15; see also Theorem 21.19), for x > 0, we have (using the abbreviation B[a,b] := {Bt : t \in [a, b]}) * P sup B[n,n+1] - Bn > x = P sup B[0,1] > x = 2 P[B1 > x] \infty 1 2 2 2 e - u/2 du$ $\leq e-x/2$. n (23.10) Proof By passing to Xi - x if necessary, we may assume E[Xi] < 0 and x = 0. Hence G = A1 \otimes A2 by Dynkin's π - λ theorem (Theorem 1.19). Define the stopping time $\tau_a, b = \inf\{t \geq 0 : Bt \in \{a, b\}\}$. Starting here, a shift in the meaning towards the mathematical notion seems plausible. Now let m > 1. A Exercise 8.3.6 (Rejection 1.19). sampling for generating random variables) Let E be a countable set and let P and Q be probability measures on E. 140 5 Moments and Laws of Large Numbers $k \rightarrow \infty$ Then (because $(1 - ak/k)k \rightarrow e - a$ if $ak \rightarrow a$) n - 1 + 2, P there is a double click in $(0, 1] = \lim P N(k 2 - n, (k+1)2 - n] \ge 2$) * $n \rightarrow \infty = 1 - \lim P n \rightarrow \infty k = 0$ n - 1 + 2 k=0 n - 1 $\lim n \to \infty$ N(k 2-n, (k+1)2-n] ≤ 1 ,)* P N(k 2-n, (k+1)2-n] ≤ 1 k=0 2n = 1 - lim 1 - P[N2-n ≥ 2] $n \to \infty = 1 - e - \lambda$. For all $j \in L$, let Aj $\in B(E)$ and Ajl := E. Then LI $\leq L$ with equality if and only if I = J. Lemma 18.3 Let X be irreducible. l=0 Hence k-1 u(k) - u(0) Reff ($0 \leftrightarrow k$) = = R(l, l + 1). Then, for $r \in Q$ and B = ($-\infty$, r], $\omega \to \kappa(\omega, B) = F(r, \omega) 1N$ $c(\omega) + F0$ (r) 1N (ω) (8.16) is F -measurable.) * 1 n < ∞ . We even have $\nu 0 \mu$. Let L := lim infn $\rightarrow \infty$ Sn. Now define, for any A \in A, $\nu a(A) := p(x, v) = p(x, v)$ y), $1{x=y}$, if $x \in A$, if (x \in A), if Therefore, $P[X1 = x1, .402 \ 17 \ Markov \ Chains \ To \ this \ end, \ let \ (qk) \ k \in N0 \ be a \ probability \ vector, \ the \ offspring \ distribution \ of \ one \ individual.$ E(Xn - Xn - 1)2 Fn - 1 = E(Yn - 1)2 Xn - 1 Fn - 1 = Var[Yn] Xn - 1 2. Show that the conditional distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X
\in B\}$ is the uniform distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X \in B\}$ is the uniform distribution of X given $\{X \in$ and let $f: \Omega \to R$ be measurable. To this end, enumerate the nodes of the graph from 1 to 12 as in Fig. Hence the value of Lx, y is unique in $\{0, . -\infty - \infty \infty) * c^* c c P(A) = P An = \lim P An$. This is the standard procedure in order to change a seminorm into a proper norm. ϕ is an infinitely divisible CFP if and only if there is a sequence (ϕ n) n \in N of the standard procedure in order to change a seminorm into a proper norm. CFPs such that $\phi(n) \to \phi(t)$ for all $t \in \mathbb{R}$. By (5.9), we also have Tkn /kn $\rightarrow E[X1]$ a.s. As in the proof of Theorem 5.16, we also get (since Yn ≥ 0) lim $l \to \infty$ Tl = E[X1] almost surely. (i) Find a sequence (fn) of elementary 3.3 functions such that fn \uparrow f. Theorem 15.32 (Moments and differentiability) Let X be a real random variable with characteristic function ϕ . We consider pN $\subset \Omega$ as an event. Then, by the Cauchy-Schwarz inequality, we have $E[|XY|] < * \ \infty$. We can thus replace f by f + f ∞ and hence assume that f ≥ 0 . Jefine (e) $\in \mathbb{N}$ for any $e \in E$ by 2–(e) $\leq pe < 2-(e)+1$. Example 8.30 Let Z1, Z2 be independent Poisson random variables with parameters $\lambda 1$, $\lambda 2 \geq 0$. Jefine (e) $\leq Pe < 2-(e)+1$. Markov property in the second equation),) * P Xt = i, Xs1 = i1, . Theorem 1.36 (Continuity and premeasure) Let μ be a content on the ring A. Hence, m solves the equation m = tanh($\beta(m + h)$). (ii) Let t \in I. Choose n \in N such that rn \in N0., X(An) are independent. 21.2 Construction and Path Properties. As we will need a similar computation for Pólya's urn model in Example 12.29, we give the details here. For a practical simulation use the computer's random number generator to produce independent random variables I1, I2, . Then: (i) F is tight \Rightarrow F is weakly relatively sequentially compact. If $\nu \in M1$ (R) with CFP $\phi\nu$ and if $\lambda > 0$, then one can easily check that $\phi(t) = -\lambda \lambda k \nu * k$. (ii) For finite families of random variables there is no perfect analog of de Finetti's theorem. An even simpler coupling can be used to show that $bm,p \leq st$ bn,p for m $\leq n$ and p $\in [0, 1]$., Xk $|| < \infty$. Clearly, ξn is an nsymmetric map. We first introduce the abstract principle of such couplings and then give some examples. 2.1 for a computer simulation of the percolation model. In the following, let E be a countable space and let p be a stochastic matrix on E. However, in the long run, we might see certain patterns. The triple (Ω , A, P) is called the Canonical Brownian motion or the Wiener space, and X is called the Canonical Brownian motion or the Viener space. process Xt := 0 f(s) dWs, $t \in [0, 1]$, is centered Gaussian with covariance function $s \wedge t Cov[Xs, Xt] = f 2(u) du$. In particular, if X is a random variable on (Ω, A, P) , then $f(x) P[X \in dx] := f(x) P[X \in dx] = f(X(\omega)) P[d\omega]$. when Φ runs through its domain. f (x, Y) $\lambda(dx)$ (ii) Let X and Y be independent and exp θ -distributed for some $\theta > 0$. Hence p p $n \to \infty$ fm - f p - 1 from do not have any further adjacent edges, this sequence of edges can be substituted by a single edge whose resistance is the sum of the resistances of the single edges (see Fig. $p(x1, x)/x \in A0$ Therefore,) * pF (x1, A0) := Px1 $\tau A0 < \tau x1 = 1$ 1 Ceff (x1 $\leftrightarrow A0$) = . Further, let U a, b = lim U-n $n \rightarrow \infty *$)) a, b * 1 + inequality (Lemma 11.3), we have E U-n $\leq b-a$ E (X0 - a); hence P U a, b < * $\infty = 1$. Note that this is not a new operation but only stresses the fact that the sets involved are mutually disjoint. Hence there is a T -measurable function ψ with $\psi = 1A$ almost surely. be i.i.d. Rad1/2 -distributed random variables (that is, P[Di = -1] = P[Di = 1] = 12 for all $i \in N$). Lemma 13.2 Let (E, τ) be a Polish space with complete metric d. Since V is complete and W is closed, W is also complete; hence there is a $y \in W$ n $\rightarrow \infty$ with wn $\rightarrow y$. Further, let μ be the stable distribution with index α whose characteristic function is given by (16.23) with c + = Cp and c - = C(1 - p). The similarity of the variables X1 X2, Then $\pi(\{x\}) > 0$ for every $x \in E$. Definition 12.11 (Backwards martingale) Let $F = (Fn)n \in -N0$ be a filtration. $n \rightarrow \infty$ Hence (Xt) $t \in [0,1]$ is a centered Gaussian process with Cov[Xs, Xt] = min(s, t). (The first inequality holds by assumption.) We infer that Yt = Xt almost surely for all t and thus (Xt) $t \in \{0,...,T\}$ is a martingale. In fact, here a probability measure μ is not uniquely determined by the values, say $\mu(\{1, 2\}) = \mu(\{2, 3\}) = 12$. We say that (Fn)n \in N converges weakly to F, for all points of continuity x of F. Hence, let $\tau := \min k \in \{1, .19.14, For this in turn it is enough to show Nn1, t \leq st Nn2, t for all t \in N0$ Then $\mu(B) = 0$ n $\rightarrow\infty$ and (fn (ω))n \in N is a Cauchy sequence in E for any $\omega \in \Omega \setminus B$. \blacklozenge Definition 1.25 (Trace of a class of sets) Let A $\subset 2\Omega$ be an arbitrary class of sets) Let A $\subset 2\Omega$ be an arbitrary class of sets) Let A $\subset 2\Omega$ be an arbitrary class of sets) Let A $\subset 2\Omega \setminus \{\emptyset\}$. generalizes the notion of product measures and points in the direction of the example from the introduction to this chapter. Thus $P[A] \in \{0, 1\}$. By choosing a suitable sequence $\Omega n \uparrow \Omega$, we can assume that ν is finite. Formally, we say that A and B are independent if $P[A \cap B] = P[A] \cdot P[B]$., $\omega n] := \{\omega \in \Omega : \omega i = \omega i \text{ for all } i = 1, . Proof We use a (0, 1) \in \{0, 1\}$. diagonal sequence argument. The corresponding network (E, C) will be called the unit network on (E, K). 18.2 Coupling and Convergence Theorem . Hint: In particular, one has to show that μ is \emptyset -continuous. Check if the logarithmic moment generating function $3 - \infty 1 + |x| \Lambda$ is continuous and sketch the graph of Λ . For $r \in Q$, let F (r, \cdot) be a version of the conditional probability $P[Y \in (-\infty, r]|F]$. We denote by E the vector space of simple functions (see Definition 1.93) on (Ω , A) and by $E + := \{f \in E : f \ge 0\}$ the cone (why this name?) of nonnegative simple functions. \bullet Exercise 21.2.6 Let B be a Brownian motion, $a \in R$, b > 0 and $\tau = \inf\{t \ge 0 : Bt = at + b\}$. (21.2) Then the following statements hold. & Exercise 14.2.3 (Partial integration) Let $F\mu$ and $F\nu$ be the distribution functions of locally finite measures μ and ν on R. \blacklozenge Theorem 4.19 Let $\mu(\Omega) < \infty$ and $1 \le p \le p \le \infty$. \blacklozenge While weak convergence of the total masses (since $1 \le cb$ (E)), with vague convergence implies convergence in the experience of the total masses (since $1 \le cb$ (E)), with vague convergence implies convergence in the experience of the total masses (since $1 \le p \le \infty$. limit. Manifestly, Xtn1, Then T (Ai)i \in I := σ Aj J \subset I #J 0. Without loss of generality, assume E = {1, Let q be the transition matrix of an arbitrary irreducible Markov chain on E (with q(x, y) = 0 for most y \in E). If $p \in$ {0, 1}, then Berq 0 Berp if and only if p = q, and Berq \perp Berp if and only if q = 1 - p. Since $\Omega \in$ D and by property (ii) of the λ system, we get that $Ac = \Omega \setminus A \in D$. + Yn $/\sigma n \Rightarrow 0$. Further, let $f: \Omega \to R$ be measurable and Xn $(\omega) = f \circ \tau n$ (ω) for all $n \in N0$., $n \} \ge t$. A content μ on A is called (i) finite if there exists a sequence of sets $\Omega 1$, $\Omega 2$, . Then (X - a) + is a submartingale. 9.4 Discrete Martingale Representation Theorem and the content μ on A is called (i) finite if $\mu(A) < \infty$ for every $A \in A$ and (ii) σ -finite if there exists a sequence of sets $\Omega 1$, $\Omega 2$, . Then (X - a) + is a submartingale. CRR Model .. (ii) The integrability condition in (i) cannot be waived. Then σ (Xi , i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai)
i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I) := σ i \in I σ (Xi) = σ Xi - 1 (Ai) i \in I i σ (Xi) = σ Xi - 1 (Ai) i \in I i σ (Xi) = σ Xi - 1 (Ai) i \in I i σ (Xi) = σ Xi - 1 (Ai) i \in I i σ (Xi) = σ Xi - 1 (Ai) i \in I i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi - 1 (Ai) i σ (Xi) = σ Xi + 1 (Ai) i σ (Xi) = σ Xi + 1 (Ai) i σ (Xi) = σ Xi + 1 (Ai) i σ (Xi) = σ Xi + 1 (Ai) i σ (Xi) = σ (Xi) = σ Xi + 1 (Ai) i σ (Xi) = σ Xi + 1 (Ai) i σ (Xi) = σ (Xi) = σ (Xi) = i (Ai) i σ (Xi) = i (Ai) i (assume F is not identically zero. The following theorem makes rigorous sense of the expressions "mainly small values" and "close to a normal distribution" and formulates the precise statement. Chapter 10 Optional Sampling Theorems In Chap. i=1 Again due to additivity and monotonicity, we get $\mu(A) = \mu \operatorname{ck} n$ ($\operatorname{Ck}_i \cap A$) = k=1 i=1 $\leq \operatorname{ck} n$ k=1 i=1 $\mu(\operatorname{Ck}_i \cap A)$ = k=1 i=1 $\leq \operatorname{ck} n$ k=1 i=1 $\mu(\operatorname{Ck}_i \cap A)$ = k=1 i u and Im(u + iv) = v denote the real part and the imaginary part, respectively, $\sqrt{of z} = u + iv \in C$. • 492 19 Markov Chains and Electrical Networks Takeaways One-dimensional random walks in a network of random walks in a network of random walks in random walks in a network of sets that are cut. Proof This is a direct consequence of (16.3). Hence we can describe the strategy by the formula Hn = 0, 2n-1, if there is an $i \in \{1, ..., Xn$ be exchangeable, square integrable random variables. drawn from a finite alphabet E (that is, from an arbitrary finite set E). Theorem 7.9 (Jensen's inequality) Let $I \subset R$ be an interval and let X be an I-valued random variable with $E[|X|] < \infty$. Proof For $n \in N$, let $\Omega = \Omega n + \Omega n - be$ a Hahn decomposition for $(\nu - n1 \mu) \in M \pm .$ Thus - k. * 15.6 Multidimensional Central Limit Theorem 365 15.6 Multidimensional (13.12) 13.3 Prohorov's Theorem 297 Then $\mu(E) \ge \sup \alpha(Kn) = \sup \lim \mu nk$ (Kn) $n \in N \ k \to \infty \ n \in N \ 1 \ge \sup \lim \sup \mu nk$ (E) $-n \ n \in N \ k \to \infty = \lim \lim \mu nk$ (E) $-n \ n \in N \ k \to \infty = \lim (n \ n \to \infty = \mathbb{N}$ (E) $-n \ n \in N \ k \to \infty = \mathbb{N}$ (E) $-n \ n \in N \ n \to \infty = \mathbb{N}$ (E) $-n \ n \in N \ n \to \infty = \mathbb{N}$ (E) $-n \ n \in N \ n \to \infty = \mathbb{N}$ (E) $-n \ n \to$ P[Bk | A] = P[A|Bk] P[Bk] P[Bk] P[Bk] P[Bk] P[Bk] A = A0. Then there exists a real d × d matrix A = (akl) with A A = A0. Then there exists a real d × d matrix A = (akl) with A A = C. Then Lp (μ) \rightarrow Lp (μ) and the canonical inclusion i : Lp (μ) \rightarrow Lp (μ), f \rightarrow f is continuous. We do not give the details but refer to [170, Chapter 14.13]. , μ in) and with $\Sigma = \Sigma$ I the diagonal matrix with entries $\sigma i 21$, However, we still have the following. Hence any predictable H is an admissible gambling strategy. Thus $X\tau-1$ (A) \in Ft . We develop it only to the point to which it is needed for our purposes: construction of measures and integrals, the Radon-Nikodym theorem and regular conditional distributions, convergence theorems for functions (Lebesgue) and measures (Prohorov), and construction of measures in product spaces. Cw (-i, -i - 1) i=1 k=-i + and R - are the effective resistances from 0 to $+\infty$ and from 0 to Note that Rw w $-\infty$, respectively. For n = 1 and f an indicator function, this is the (time-homogeneous) Markov property., 6}3 endowed with the discrete σ -algebra A = 2 Ω and the uniform distribution P = U Ω (see Example 1.30(ii)). To this end, we can assume p = p*. Let K > 0 and Yn := Xn 1{|Xn| \leq K} for all n \in N. In (iii) one can even show that almost surely there are infinitely many infinite connected components., pk \in P, we have ki=1 (pi N) = V Ω (see Example 1.30(ii)). To this end, we can assume p = p*. Let K > 0 and Yn := Xn 1{|Xn| \leq K} for all n \in N. In (iii) one can even show that almost surely there are infinitely many infinite connected components. $(p_1 \cdots p_k)$ N. Furthermore, E[Xi-1 Xi Fi-1] = Xi-1 E[Xi Fi-1] = Xi-1 E[Xi Fi-1] = Xi-1 i=1 = n + E[Xi Fi-1] = 2; hence (as in (10.1)) Xi-1 n = 2 = 1; hence (as in (10.1)) Xi-1 n = 2; hence (as in (10.1 6.13, there is a k $\rightarrow\infty$ subsequence (fnk)k \in N of (fnk)k \in N with fnk $\rightarrow f$ almost everywhere. (0 $\leftrightarrow \infty$) = ∞ . Show that PA Xi i \in I is uniformly integrable. \blacklozenge Definition 1.6 A class of sets A $\subset 2\Omega$ is called an algebra if the following three conditions are fulfilled: (i) $\Omega \in A$. Thus, by Corollary 1.82, XJI is measurable. However, the proof is a little more involved. Note that $kn \ge \alpha n/2$. Let $p^{\sim}e = 2-(e)$ for any $e \in E$ and let $q^{\sim}k = 1 k$; hence for all l > k. m=1 In particular, for every $f \in H$, Parseval's equation $f^{2} = \infty$)f, bm * 2(21.22))f, bm * 2($(A) := \nu(A \setminus E)$ and $\nu s(A) := \nu(A \cap E)$. Otherwise the proposal is accepted only with probability $\pi(x i)/\pi(x)$. 1.3 The Measure Extension Theorem In this section, we construct measures μ on σ -algebras. Xti – Xti lim E $n \rightarrow \infty$ i=1
$n \rightarrow \infty$ In particular, Xtn1, Proof As A is a π -system, uniqueness follows by Lemma 1.42. Clearly, the value of Hn has to be decided at time n - 1; that is, before the result of Xn is known. Proof For the existence of $E[\phi(X)|F]$ with values in $(-\infty, \infty]$ note that $\phi(X) - \in L1$ (P) and see Remark 8.16. 19.16 Crossed ladder graph z a Fig. 474 19 Markov Chains and Electrical Networks Let u = ux1 , A0 be the unique potential function on E with u(x1) = 1 and u(x) = 0 for any $x \in A0$. In particular, $\{x\} \in B(Rn)$ for every $x \in Rn$. Example 13.20 If X, X1, X2, For this generalization of the conditional expectation, we still have $E[X | F] \le E[Y | F]$ a.s. if $Y \ge X$ a.s. (see Exercise 8.2.1). Summing up, we get # qn ([0, 1)) $\le 2n$. Hence, let 0 = t0 < t1 < . In probability theory, in the second category fall quantities such as the median, mean and variance of random variables. Then $f \in L1$ (μ) and $fn \rightarrow f 3 3 n \rightarrow \infty$ in L1; hence in particular fn $d\mu \rightarrow f d\mu$. We will exploit the following criterion for tightness of M1 (M1 (E)). This process can be regarded as a Galton-Watson branching process in continuous time. i=1 Letting $\epsilon \downarrow 0$ implies (4.2) and hence the claim (ii). we denote the path that starts with one step in direction x, then chooses y, then x, then z and so on. (19.17) This is the triangle inequality for the effective resistances and it shows that the effective resistance is a metric in any electrical network. For $N \in N$ and 0 = t0 < t1 < . Exercise 17.6.6 Let X be irreducible and recurrent. (ii) Show that the effective resistance is a metric in any electrical network. return to 0 is M and infer that the chain is positive recurrent if and only if M < ∞. Proof F1 is continuous with respect to d and hence B(Ω, d)-measurable. The most important properties of weak and vague convergence, E1 supn∈N τ is, in finite time, X exceeds all levels. Since $E_j \cup \{\emptyset\}$ is a π -system, Lemma 1.42 yields that $\mu(E_j) = \nu(E_j)$ for all $E_j \in \sigma(E_j)$. Recall that $B\epsilon(x) = \{y \in E : d(x, y) < \epsilon\}$ denotes the open ball of radius $\epsilon > 0$ that is centered at $x \in E$. (Note that this is not a stopping time.) By the Markov property, for all $k \le n$,)) * P $\pi \sigma xn = k = P\pi Xk = x$, Xk+1 = x, . 21.1 Continuous Versions A priori the paths of a canonical process are of course not continuous since every map $[0, \infty) \rightarrow R$ is possible. At this point we use Lemma 7.46. +s - Ts no 2 In either case, we have \sqrt{Kn} , n t n Kn Kn Kn , n T⁻ - T | , $\leq |Yk| + |Yk+1t+s \circ \sigma$ hence ' 2 4 ((n, n T Kn Kn Kn | 2 E T K +s - Ts 12 4 or Continuous Since every map $[0, \infty) \rightarrow R$ is possible. At this point we use Lemma 7.46. +s - Ts no 2 In either case, we have \sqrt{Kn} , n t n Kn Kn Kn Kn , n T⁻ - T | , $\leq |Yk| + |Yk+1t+s \circ \sigma$ hence ' 2 4 ((n, n T Kn Kn Kn | 2 E T K +s - Ts 12 4 or Continuous Since every map $[0, \infty) \rightarrow R$ is possible. At this point we use Lemma 7.46. +s - Ts no 2 In either case, we have \sqrt{Kn} , n t n Kn Kn Kn Kn + \sqrt{Kn} +) * 16n5/2t 4 16 \leq Var Y1Kn \leq 2 t 3/2., Xn be independent exponentially distributed random variables with parameters $\theta 1$, 428 17 Markov Chains If $\mu = 0$, then μ is a finite measure if and only if $M := \infty n-1$ pk $< \infty$., ∞ . be i.i.d. Define Sn = X1 + . By Remark 6.3, it is enough to consider the case $\mu(\Omega) < \infty$. Then the stochastic kernel (x, A) $\rightarrow \kappa Y, X$ $(x, A) = P[\{Y \in A\}|X = x] = \kappa Y, \sigma(X) X - 1 (x), A$ (the function from the factorization lemma with an arbitrary value for $x \in X(\Omega)$) is called a regular conditional distribution of Y given X. Together with (12.8), we get (12.7). Takeaways If a random variable is the sum of many independent centred random variables, each of which takes mainly small values, then its distribution is close to a normal distribution., N - 1, $r \in (0, 1)$ and $p(i, j) = \int \{r, 1 - r, [0, if j = i + 1 \pmod{N}, else, g + (f + -g +) + d\mu = 100 4 \text{ The Integral Similarly}, we use <math>f - \geq g - a.e.$ to obtain $f - d\mu \geq g - d\mu$. Proof One implication is trivial. Thus $X + Y \sim Poi\lambda + \mu$. If in the ith game the player makes a (random) stake of Hi euros, then the cumulative profit after the nth game is Sn = n Hi Di . Let F1 and F2 be the distribution functions of μ 1 and μ 2 . , Xn) \in G] = 1. Then f is B(Ω) – B(Ω)-measurable. Then ν can be uniquely decomposed into an absolutely continuous part ν a and a singular part ν s (with respect to μ): $\nu = \nu a + \nu s$, where $\nu a 0 \mu$ and ν s \perp μ . \bullet 0 1 x Exercise 19.5.3 Consider the graph of Fig. We ignore the spatial structure and assume that any particle interacts with the fact that X is not uniformly integrable. k k=0 Using this formula with x = (1 - p) eit gives the claim. \bullet Exercise 13.1.9 Show that the set of rationals Q (with the standard topology) is not a Polish space. 370 16 Infinitely Divisible Distributions Theorem 16.5 A probability measure μ on R is infinitely divisible if and only if $n \rightarrow \infty$ there is a sequence (νn) n \in N in Mf (R \ {0}) such that CPoi $\nu n \rightarrow \mu$. The length l(e) of the sequence that codes for e may depend on e. Then there is an $\varepsilon > 0$ and a subsequence (fnk) $k \in N$ with $f - fnk 1 > 2\varepsilon$ for all $k \in N$., xn) $\in E$ n, denote x = (x(1), . By assumption, P[$\tau K = \infty$] $\rightarrow 1$ for $K \rightarrow \infty$; hence X converges almost surely. Since X converges almost surely. some s > 0. inf $0 \le g \in L1$ (µ) f $\in F$ (ii) inf sup (|f| - a) + dµ = 0, a \in [0, \infty) f $\in F$ (ii) inf sup (|f| - a) + dµ = 0, a \in [0, \infty) f $\in F$ (ii) inf sup (|f| - a) + dµ = 0, a \in [0, \infty) f $\in F$ (ii) inf sup (|f| - a) + dµ = 0, a \in [0, \infty) f $\in F$ (ii) inf sup (|f| - a) + dµ = 0. (A \cap Bi) = i \in I P[A \cap Bi] = P[A|Bi]P[Bi]. However, in general, this property is weaker than (i) in Definition 1.6. Theorem 1.7 A class of sets A $C 2\Omega$ is an algebra if and only if the following three properties hold: (i) $\Omega \in A$. Hence $An \in Ai$ for every $n \in N$ and $i \in I$., n normal distribution $\Gamma\theta$, r on $[0, \infty)$ Beta distribution $\Sigma\theta$, r on $[0, \infty)$ Beta distribution $\Sigma\theta$, r on $[0, \infty)$ Beta distribution $\Gamma\theta$, r on $[0, \infty)$ Beta distribution $\Sigma\theta$, (1.17) by combik (-1) = k natorial means from its interpretation as a waiting time. $n-2 \le 4$ for all $x \ge 0$. By Remark 6.3, (fnkl) l \in N converges to f almost everywhere. 9 we saw that martingales if we apply certain admissible gambling strategies. .) \in E N . 21.10 Square Variation and Local Martingales ..., 2n-1, let necessarily continuously) differentiable at 0. The Feller property and Theorem 21.24 ensure the existence of an RCLL version q of Xq. Example 17.60 Let $n \in N$ and $0 \le p1 \le p2 \le 1$. Proof In Theorem 17.9, let ti = i for every $i \in N0$. Proof This is a consequence of Theorem 6.25, as the dominating function ensures uniform integrability of the sequence of the sequence of Theorem 6.25, as the dominating function ensures uniform integrability of the sequence of the sequenc $(fn)n \in N$., pm) be a probability vector on $\{1, . The star-shaped part of a network (left in Fig. In fact <math>\sigma$ (($-\infty, a$], $a \in Rd$) = B(Rd) by Theorem 1.23. for all $z \in [0, \eta)$ for some $\eta \in (0, 1)$. However, we have just shown that such a path could not exist. In other words: For any simple event $\omega \in \Omega$, Xn (ω) yields the result of the nth experiment. In this case, we also write $\nu = f\mu$ and $f = d\nu$, i=1 Can we extend the set function μ to a (uniquely determined) measure on the Borel σ -algebra B(Rn) = σ (A)? \bullet Takeaways Assume that the price of a risky asset at discrete trading times n = 0, 1, 2, . n=1 n=1 (ii) For any $A \in \sigma$ (A) with $\mu(A) < \infty$ and any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ and n mutually disjoint sets A1, t that $\phi = (\phi \ 1/n \)n$ is infinitely divisible. 23.1 Cramér's Theorem . Letting $\phi k := 1Ck$, Theorem 12.17 implies that $(\ 1A = E[1A | E] = E \ lim \ E[\phi kl \ (X)|T] =: \psi \ l \rightarrow \infty \ lmost \ surely$. Proof We compute $\psi S(z) = \infty \ P[S = k] \ zk \ k=0 = \infty \ \infty \ P[T = n] \ P[X1 + . Exercise 5.3.1 \ Show \ lmost \ surely$. improvement of Theorem 5.16: If X1, X2, . (ii) If E is locally compact, then in ((i)) we can choose the neighborhoods Ux to be relatively compact. However, since f is continuous, we even have f -1 (A) $\in \tau$ 1.7.4. Definition 7.19 Let V be a real vector space. 2. Thus X would not be irreducible contradicting the assumption. dx [0,x] 13.2 Weak and Vague Convergence 281 Hint: Use Exercise 13.1.6 with μq (dx) = (f (x) - q) + λd (dx) for $q \in Q$, as well as the inequality μq (x + rC) $\mu (x + rC) \neq (q + . 25.1 \text{ Itô Integral with Respect to Brownian Motion}$., tn) for x = (x1, . Since $\mu(Z) = \phi(0) = 1$, μ is indeed a probability measure. Then there exists an F^{∞} -measurable random $n \rightarrow \infty$ variable X^{∞} with $E[|X \propto |] < \infty$ and $Xn \to X \propto$ almost surely. The following statements are equivalent. 2α Let X1, X2, this, we construct a subgraph for which we can compute
Reff Sketch We consider the set of all infinite paths starting at 0 and that • begin by taking one step in the x-direction, the y-direction, • continue by choosing a possibly $n \in N$ $n \in N$ $\rho \in C$. (iii) Every algebra is a ring. However, here we sketch a particular situation where this is a ring. However, here we sketch a particular situation where this is a ring. However, here we sketch a particular situation where this is possible. with $A \subset i=1$ i=1 Definition 1.28 Let A be a semiring and let $\mu: A \rightarrow [0, \infty]$ be a set function with $\mu(\emptyset) = 0$. By (i), we get $E[X\tau \land n] \leq E[X\sigma]$ for any $n \in N$. Define $P\pi = x \in E \pi(\{x\})Px$. Since μ is inner regular (Theorem 13.6), there is a compact set $K \subset G$ with $\mu(G) - \mu(K) < \epsilon$. Summing up, we have '2 (*) E(X - Y)2 - EX - E[X|F] (' = EX^2) -2XY + Y2 - X2 + 2XE[X | F] - E[X | F] - E[X | F] - E[X | F] + E[X | F] + E[X | F] + E[X | F] + E[X | F] - E[X | F] - E[X | F] - E[X | F] + E[X | F] - E[X | F] - E[X | F] + E[X | F] - E[X | F] + E[X | F] - E[X | F] - E[X | F] + E[X | F] - E[X | F] - E[X | F] - E[X | F] + E[X | F] - E[X | F] - E[X | F] + E[X | F] + E[X | F] - E[X | F] + E $E[|Xn - X|] \leq E[Zn] - \rightarrow 0$. (ii) Define independent random variables Xn, i, Yn, i, n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and \Theta n, n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i = 1, 2, and Θn , n $\in N$, i $= x \in E \pi({x})Px$. (ii) Let $A = {\omega : f(\omega) = \infty}$. For the latter case, the claim of the theorem would be incorrect since, loosely speaking, with H we can bet on X but not on the Di . 452 18 Convergence of Markov Chains The Gibbs sampler for the Ising model is thus the Markov chain (Xn)n \in N0 with values in $E = {-1, 1}A$ and with transition matrix $\int \frac{1}{2\pi \pi} e^{-1} e^{-1$ $(-1 \{ 1 \mid + \exp 2\beta \ (1 \{ x(j) = x(i) \} - 1), if y = x i \text{ for some } i \in \Lambda, 2 p(x, y) = \#\Lambda j : j \sim i \{ 0, otherwise. \blacklozenge The Stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The measure is the characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies that a characteristic function determines a probability distribution uniquely. \blacklozenge The stone-Weierstraß theorem implies the stone-Weierstraß theorem implies the stone-Weierstraß theorem implies the stone-Weierstraß theorem implies theorem implies theorem implies$ a < b is called the Lebesgue-Stieltjes measure with distribution function F. Similarly, there exist $dk \in N$ and Dk,1, . Recall that a σ -algebra I is called P-trivial if $P[A] \in \{0, 1\}$ for every $A \in I$. Define a, b UNa, b = UQa, b + $\cap[0,N]$., Xn $\} = Y1 + .$ Then $\mu *$ is an outer measure. Definition 20.5 An event $A \in A$ is called invariant if $\tau - 1$ (A) = A and quasiinvariant if $1\tau - 1$ (A) = 1A P-a.s. Denote the σ -algebra of invariant events by I = A \in A : $\tau - 1$ (A) = A ., Xk] = E[F (Xi1 , . & Exercise 8.2.7 Let X1 , . Of course, any of the pairs (A1 , A2), (A1 , A3) and (A2 , A3) has to be independent. We will also assume that (E, τ) is a Hausdorff space; that is, for any two points $x, y \in E$ with x = y, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. Define $\mu 1 = \mu(E1 \cap \cdot)$ and $\mu n = \mu((En \setminus En-1) \cap \cdot)$ for $n \ge 2$. 16.1 Lévy-Khinchin Formula 369 As convolutions of Poisson distributions play a special role, we will consider them separately. Proof This is a consequence of Theorem 24.5 and the uniqueness theorem for characteristic functions (Theorem 15.9) and for Laplace transforms (Exercise 15.1.2) of random variables on $[0, \infty)n$. Therefore, X does not even converge improperly. 461 462 465 467 473 480 488 20 Ergodic Theory. We consider a strengthening of (20.7). Note that t $\rightarrow \Lambda(t)$ is differentiable (with derivative Λ) and is strictly convex. Here we used the exchangeability of X in the first equality and the symmetry of F in the second equality. Let sk = x1 + . Hence, there exist $N \in N$ and $y0 \leq -f \propto < y1 < .$ (ii) Let $\gamma \in 0$, $\beta \alpha . i=1$ 1.1 Classes of Sets 9 (i) There are subsets of Rn that are not Borel sets. (16.23) \blacklozenge 2 Lemma 16.24 Let μ be infinitely divisible with 3 canonical triple (σ , b, ν); that is, it x with log-characteristic function $\psi(t) := \log e \mu(dx)$ given by $\sigma^2 \psi(t) = -t 2 + ibt + 2$ it $x e - 1 - itx 1 \{|x| 0, d \in \mathbb{R}, n \in \mathbb{N} \text{ and let } X, X1, . "(iii) \Rightarrow (i)"$ Assume that (i) does not hold. (ii) If in addition A is $\sigma - 0$ -closed, then A is $\sigma - 0$ -closed. Show the following statements: 2 (i) $\infty n = 0$ An $< \infty$ almost surely. (ii) Does the converse implication hold in (i)? By choosing the minimal $n \ge n0$ such that $|t - s| \ge 2 - n$, we obtain by (21.8), $|Xt(\omega) - Xs(\omega)| \le C0 |t - s|\gamma$. A Proof (i) Let f be the map $f: \Omega \to I$ with $f(\omega) = i \iff Bi \omega$. (Result: $\Lambda * (x) = \theta x - \log(\theta x) - 1$ if x > 0 and $= \infty$ otherwise.) & Exercise 23.2.5 Compute Λ and $\Lambda *$ for the case where X1 is Cauchy distributed and interpret the statement of Theorem $23.11. Q \pm (A) = A \text{ Now define } Y = Y + -Y - .22, 24, \text{ and } 25, 26 \text{ can be read independently., xn}$. We conclude that (since P[X2j = 0] = and P[X2j + 1 = 0] = 0, 2j - j 4. Indeed, ∞ the improper integral 0 f (x) dx := limn \rightarrow \infty 0 f (x) dx is defined by a limit procedure
that respects the geometry of R. \bullet Example 12.3 (i) If (Xi)i \in I is i.i.d., then (Xi)i \in I is exchangeable. (24.2) Proof The class of sets I = (If1, . Then M = B(τv) = σ If : f \in Cc (E) = σ If : f (F \cap G) (F (F \cap G)) = σ If : f (F \cap G) (F (F \cap G)) = σ If : f (F \cap G) (F (F \cap G)) = σ If : f (F \cap G) (F (F \cap G)) = σ If (F \cap G) (F (F \cap G $= (Xt) t \in [0,1]$ that is defined by the covariance function Γ (s, t) = s \wedge t - st. Remark 5.31 Condition (5.15) is sharp in the sense that for any increasing sequence of pairwise independent, square integrable, centered random variables X1, X2, However, if F1n turns out to be a constant map (e.g., $F1n \equiv x *$ for some random x *), then we will also have $F1m \equiv x *$ for all $m \ge n$. n=1 Clearly, C is a countable set of compact sets in E, and C is stable under formation of unions. Inductively, we get (finite) additivity. As an application, we (Z_{\epsilon}) e get the statistical physics principle of minimising the free energy. (iii) X has the time homogeneous Markov property (MP): For every $A \in B(E)$, every $x \in E$ and all s, $t \in I$, we have *) Px Xt + s $\in A$ Fs = κt (Xs, A) Px -a.s. Here, for every $t \in I$, the transition kernel defined for $x \in E$ and $A \in B(E)$ by $\kappa t (x, A) := \kappa x$, $\{y \in E \mid y \in I : y(t) \in A\} = Px [Xt \in A]$. Proof Let $f : E \rightarrow R$ be bounded and Lipschitz continuous with constant K. Lemma 15.37 Let X1, X2, Inductively, we get n κk (0, ϕn (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = \kappa k \phi n-1 (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = \kappa k \phi n-1 (A1 ×···× An)) k=1 n-1 = $\kappa k \phi n-1$ (A1 ×···× An)) k=1 n-1 = \kappa k \phi n-1 (A1 edges in $E \setminus E$ p are called closed. Similarly, we can use the first inequality in Lemma 23.12 to get *) $P \xi n(X) \in A \ge -$ lim sup inf $I \mu (A \cap En)$. With this interpretation, it is particularly evident that it is natural to construct new stochastic processes by considering investment strategies for the stock. Proof Let ϕ be the CFP of μ . n (iii) k=0 Ak, n \in N converges almost surely. \in A with An \uparrow A. 154 6 Convergence Theorems By construction, 3 {|f|>g ϵ } ; hence also {|f|>g} ; h Mf ($[0, \infty)$). Let I \subset R be an interval. n 20.2 Ergodic Theorems 499 Let $\epsilon > 0$ and F := {Z > ϵ }. k=1 Hence μ is subadditive. In particular, letting c = f - gp we obtain p 1/p f - gp = (1 + $\mu(\Omega)$)1/p f - gp . However, we have to make sure also that the simultaneous occurrence of A1 and A2 does not change the probability that A3 occurs. Then PX =: Poi λ is called the Poisson distribution with parameter λ . For all $t \ge 0$, the function (compare (15.8)) ft : $[0, \infty) \rightarrow [0, \infty)$, $x \rightarrow 1 - e - tx$, if x > 0, if x = 0, 374 16 Infinitely Divisible Distributions is continuous and bounded (by t v 1). (2.17) This part of the proof is the most difficult one. Accordingly, assume F is uniformly integrable (but not necessarily $\mu(\Omega) < \infty$). 16.2, we will show that this is true for all $\alpha \in (0, 2]$. For measurable $g : \Omega \to \mathbb{R}$, let Ug be the set of points of discontinuity of g. By the Hahn-Banach theorem $n \to \infty$ of functional analysis (see, e.g., [87] or [174]), F can be extended to a continuous linear functional on ∞ . By construction, X is stationary with respect to P lim P and $\leftarrow -n \rightarrow \infty$ n) $n \in N0 - 1 = P \circ (Xn) n \in N0 - 1$. On the other hand, each Xi is continuous and thus measurable with respect to B - Bi. That is, we obtain the following lemma. n n -1 n - 1 k = 0 k = 0 n $\rightarrow \infty$ By Birkhoff's ergodic theorem, we have Yn $- \rightarrow P[B]$ almost surely. Similarly, the claim follows for (IA1, . As a by-product we get a discrete version of the celebrated Black-Scholes formula. Proof The space L2 (µ) fulfills the conditions of Corollary 7.27. Then use the approximation arguments as in Exercise 4.2.5. 4.3 Lebesgue Integral Versus Riemann Borel- n=1 Cantelli lemma (Theorem 2.7), we infer lim sup $n \rightarrow \infty$ 1 sup Bt – Bn, t \in [n, n + 1] = 0 n Hence X is also continuous at 0. 19.6)., BtN). (14.5) i=1 n μ i := μ 1 $\otimes \cdots \otimes \mu$ n := μ is called the product measure of the μ i . Proof Assume that there does exist a topology that induces almost everywhere meas convergence. * wn n=0 17.4 Discrete Markov Chains: Recurrence and Transience In the following, let X = (Xn) n endow walks. (x) Somewhat more generally, there is no nontrivial infinitely divisible distribution that is concentrated on a bounded interval. TV ; Thus, for every $x \in E$, we have lim sup ; $\delta x \text{ pn} - \pi$; TV > 0. $\in M \leq 1$ (E). If $x \in E$, then denote by $x i, \sigma$ (j) = i, x(j), if j = i, x(j), y(j), y(j(Fig. 13.1 A Topology Primer 3 Here come the details. $x \ge 0$, $\lambda \in I$ dh $x \ge 0$, $\lambda \in I$ dh $x \ge 0$, $\lambda \in I$ dh $x \ge 0$, $\lambda \in I$ a successful coupling and Xn = Yn for $n \ge \tau$. The set of such rectangular cylinders for which in addition Aj \in E for all $j \in J$ holds will be denoted by ZJE, R. 17.5 Application: Recurrence and Transience of Random Walks 419 Hence we only have to compute the asymptotics of P0 [Yt1 = 0] for large t. Therefore, (iii) does $n \rightarrow \infty$ not hold. Under certain assumptions on the continuity of the paths, however, the two notions coincide. We assume that there exists a map $\psi: [0, \infty) \rightarrow [0, 1]$ n $\rightarrow \infty$ that is continuous in 0 and such that $\psi n \rightarrow \psi$ pointwise. (i) Let $p \in [0, 1]$ and P[X = 0] = 1 - p. Show that the following hold. The final step is to show convergence in path space. Furthermore, we have pf = f on $E \setminus A$. 416 17 Markov Chains j Clearly, Y1 = (Y11, .) Proof " \leftarrow " This is continuous in 0 and such that $\psi n \rightarrow \psi$ pointwise. (i) Let $p \in [0, 1]$ and P[X = 0] = 1 - p. Show that the following hold. The final step is to show convergence in path space. obvious. $n \rightarrow \infty$ Proof Let (fN)N \in N be a sequence in Cc (E; [0, 1]) with fN \uparrow 1. This is Donsker's theorem that is known in the physics literature as the invariance principle. $\epsilon \rightarrow 0$ Note that $\rho C, \epsilon \rightarrow 1C$. Furthermore, $P[A \cap B] = \tilde{H}(A^* \times B) \# A^* \# B^* = \cdot = P[A] \cdot P[B]$. Our intuitive understanding of an edge is a connection between two points x and y and not an (unordered) pair $\{x, y\}$. Note that R(x) > 0,
as A is open. As we want to use only zeros and ones (and no gap-like symbols), we have to arrange the code in such a way that no code is the beginning of the code of a different symbol. If, on the other hand, (Nt, t ≥ 0) is a Poisson process, then (Nt – Ns, (s, t] $\in I$) has properties (P1)-(P5). (ii) Consider the Bernoulli distributions Berp and Berq for p, $q \in [0, 1]$. Define u(0) = 0 and u(2) = 1.11.2 Martingale Convergence Theorems .. One can use second moments and Chebyshev's inequality to establish a weak law of large numbers. For extensive literature on stochastic aspects of mathematical finance, we refer to the textbooks [9, 42, 48, 57] 86, 102, 121] or [160]. = 2 and Var[X] < ∞ , let E[X] = 0. Unlike a, of s depends on every coordinate if it is finite. (ii) If F, G C L1 (μ) are uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F} are also uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F} are also uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F} are also uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F} are also uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F} are also uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F} are also uniformly integrable. Definition 9.10 A stochastic process X = (Xt, t \in I) is called adapted to the filtration F if Xt is Ft \cdot g \in G) and {|f|: f \in F \cdot g \in G} and {|f|: f \in F \cdot g \in G} and {|f|: f \in F \cdot g \in G} and {|f|: f \in F \cdot g \in measurable for all t \in I. For probability measures on R, weak convergence is tantamount to convergence of distribution functions at all points of continuity of the limiting function. ***** 488 19 Markov Chains and Electrical Networks a z Fig., ω = $|\omega(0)| \leq K\epsilon$, V (ω , $\delta N, k, \epsilon$) $\leq k$ 548 21 Brownian Motion By the Arzelà-Ascoli theorem, $C\epsilon := CN,\epsilon$ is relatively compact in $C([0, \infty))$ N \in N and we have Pi ($C\epsilon c$) $\leq \infty \epsilon + Pi \omega : V N (\omega, \delta N, k, \epsilon) > 1/k \leq \epsilon 2$ for all $i \in I$. In the special cases f(x) = 2 and $f(x) = 2 \cdot 1\{0\}$ (x), respectively. With the notation of Theorem 1.53, this completion is Ω , $M(\mu *), \mu * M(\mu *)$. *b - = b - Exercise 3.1.1 Show that br, p r+s, p for r, s $\in (0, \infty)$ and p $\in (0, 1]$. 2.1 Independence of Events 57 Proof This is left as an exercise. \blacklozenge The following two definitions make sense also for more general index sets I that are partially ordered. Existence Let X + = X v 0 and X - = X + - X. (iii) Since $|\alpha f + \beta g| \leq |\alpha| \cdot |f| + |\beta| \cdot |g|$, Lemma 4.6(i) and (iii) yield that $\alpha f + \beta g \in L1(\mu)$. }) + $\mu(\{N + 1, N + 2, . be distribution functions of probability mean \rightarrow \infty sures on R. For B \in A, P[B | F] := E[1B | F] is called a conditional probability of B given the <math>\sigma$ -algebra F. By k=1 n $\rightarrow \infty$ n $\rightarrow \infty$ (ii) Clearly, $\phi Y(t) = \infty *$) P[N = n] E ei)t, X1 + ... + Xn * n = 0 = ∞ P[N = n] $\phi X(t)$. Rather, we just used the fact that the differences $(\Delta X)n := Xn - Xn - 1$ take only the values -1 and +1. Hence t = 0 X is the required Markov process., $\alpha N \in (0, \infty)$ and mutually disjoint sets A1, Therefore, by Theorem 19.25, is finite if and only if Rw w X is transient $\Rightarrow - + < \infty$ or Rw $< \infty$. Now assume that a prefix code is given. Indeed, $\phi r, p(t) = er\psi p(t)$, where $(br/n, p \psi p(t))$ $= \log(p) - \log 1 - (1 - p)$ eit. Since $\{A\} \in U(A)$, we have $\mu(A) \le \mu(A)$. rk! k - = CPoi If we had br, p rv for some $v \in Mf(N)$, then we would have $\nu(\{k\}) = k (1 - p)/k$. We conclude that Nn1 ,t +1 \le st Nn2 ,t +1 which completes the induction and the proof of the theorem. Use the martingale convergence theorem to n=1 n2 show the strong law of large $\nu \in Mf(N)$, then we would have $\nu(\{k\}) = k (1 - p)/k$. We conclude that Nn1 ,t +1 \le st Nn2 ,t +1 which completes the induction and the proof of the theorem. Use the martingale convergence theorem to n=1 n2 show the strong law of large $\nu \in Mf(N)$. numbers for (Xn)n \in N. Definition 1.46 (Outer measure) A set function $\mu * : 2\Omega \rightarrow [0, \infty]$ is called an outer measure if (i) $\mu * is \sigma$ -subadditive. We denote the effective conductance from x1 to ∞ by Ceff (x1 $\leftrightarrow \infty$) := C(x1) inf pF (x1 , A0) : A0 \subset E with $|E \setminus A0| < \infty$, A0 x1. That is, Z \in Z is a finite subset of A such that the sets $C \in Z$ are pairwise disjoint and $C \in Z$ $C = \Omega$ for all Z. Thus also $Ac \in Ai$ for any $i \in I$. (5-12) Exhaustion arguments similar to that in (4) also work for rectangles. \bigstar Example 19.28 Symmetric simple random walk on E = Z2 is recurrent. Let L > 0 and let (Xn) $n \in N$ be a martingale with the property Xn+1 - Xn $\leq L$ a.s. (11.2) Define the events $C := (Xn) n \in N$ converges as $n \to \infty$, $A + := \lim sup Xn < \infty$, $n \to \infty$, $n \to$ here, we leave its proof as an exercise (compare Exercise 17.6.6; see also [39, Section 6.5]). For example, such a process can be used to describe the motion of a particle immersed in water or the change of prices in the stock market. 260 12 Backwards Martingales and Exchangeability (iii) For x ∈ E N, define the nth empirical distribution by ξn (x) = n1 ni=1 δxi (recall that δxi is the Dirac measure at the point xi). In this case, kn is called the family of n-step transition probabilities. The maximal entropy of a probability measure on N points is achieved by the uniform distribution and is log(N). 15.2 illustrate this. *1/n Proof Apply Theorem 16.6, where ϕn is the CFP of μn . In many cases, the lower bound is a lot easier to show than the upper bound. 482 19 Markov Chains and Electrical Networks Step-by-Step Reduction of the systematic computation of these effective resistances. At first reading, the reader might wish to skip this rather analytically flavored chapter. Here the statement is trivial (choose a = 0, b = 1 and c = -E[Y]). Due to the analogy of (19.3) to Green's formula in continuous $\tilde{}$ is called the Green function for the equation space potential theory, the function G (p - I) f = 0 on $E \setminus A$. Define p(x, y) = 1 $q(x, y) + I(x, y) \lambda$ for $x, y \in E$, if $\lambda > 0$ and p = I otherwise. For $n \in N0$, define $YnN := Yn . In particular, a probability measure \mu on R is uniquely determined by its distribution function F : R \rightarrow [0, 1], x \rightarrow \mu((-\infty, x]). That is, (2.8) holds with J replaced by J \cup \{j\}. For n \in N choose \mu n \in M1 (R) n such that \mu * n n = \mu and let \phi n the CFP of \mu n . \in A with (a, b] \subset \infty (a(k), b(k)]. Denote by \mu 0n \in M1 (M1 (\Sigma)) the corresponding a priori$ distribution of x; that is, of the n-particle system. Since $1 \wedge x \leq 2(1 - e - x)$ for all $x \geq 0$, clearly $(1 \wedge x) \nu(dx) \leq 2(1 - e - x) \nu(dx) \leq 2(1 - e - x)$ for all $x \geq 0$, clearly $(1 \wedge x) \nu(dx) \leq 2(1 - e - x) \nu(dx) = 2(1 - e - x) \nu(dx) \geq 2(1 - e - x) \nu(dx) \geq 2(1 - e - x) \nu(dx) = 2(1 - e - x) \nu(d$ + $g d\mu 2 - f d\mu 2 \le \varepsilon (\mu 1 (E) + \mu 2 (E))$. It remains to show that PNt = Poi α t. n + 1 - ns Proof Compute $\psi(s) = \infty 2 - k - 1$ s k = k=0 1. Hence, the random walk X = (Xn) n \in N sup Sup Fn (x) - F (x) = 0 almost surely. One can even construct infinite products if all factors are probability spaces (Theorem 14.39). That is, two points x, $y \in E$ are connected by an edge if they differ in exactly one coordinate. By Fatou's lemma, lim sup $gk d\mu = k \rightarrow \infty g d\mu - \lim (g - gk) d\mu \le g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g d\mu
- \lim (g - gk) d\mu = g d\mu - \lim (g - gk) d\mu = g$ $r \rightarrow 0$ and thus the semigroup is continuous. F is called right continuous if F (x) = limn \rightarrow \infty F (xn) for all $x \in Rd$ and every sequence (xn) n $\in N$ in Rd with $x_1 \ge x_2 \ge x_3 \ge .$ be identically distributed. This shows (ii). Then Y is manifestly a σ (X)-martingale. $k \rightarrow \infty$ (13.11) Assume that we can show that there is a measure μ on the Borel σ -algebra E of E such that $\mu(A) = \sup \alpha(C) : C \in C$ with $C \subset A$ for all $A \subset E$ open. meas With our definition of stochastic convergence we have fn $- \rightarrow f$. Since μ is indeed a (possibly defective) distribution function if μ is a (sub-) probability measure. Clearly, $\sigma(E) = 2\Omega$ but E is not a π -system. j $\in \Lambda$: i \sim j Here i \sim j indicates that i and j are neighbors in Λ (that is, coordinate-wise mod N, we also speak of periodic boundary conditions). This implies lim sup P[A $\cap \tau - n$ (B)] – P[A] P[B] |n| $\rightarrow \infty \leq \lim \sup P[A \epsilon \cap \tau - n (B \epsilon)] - P[A \epsilon] P[B \epsilon] + 4\epsilon = 4\epsilon$. In particular, F and σ (X) are independent. Define Mn = max{0, S1, Proof The proof makes use of the Radon-Nikodym theorem (Corollary 7.34). If c - = c + d, then μ is a Cauchy distribution. We now take the waiting times as the starting point and, based on them, construct the Poisson process. Hence, for $n \in N0$, pl+n+k (y, y) $\geq pl$ (y, x) pn (x, y) k (x, y). Exercise 9.2.2 Let (Xn) n \in N0 be a predictable F-martingale. As (Zn) n (x) pk (x, y) $\geq pl$ (y, x) pn (x, y) k (x, y). $(E[Zn \ F])n\in N$ decreases to some limit, say, Z. (ii) Let $E(\omega)$ be a property that a point $\omega \in \Omega$ can have or not have. The procedure of defining two families of random variables that are related in a specific way (here "<") on one probability space is called a coupling. / Then (Ω, τ) is Polish and B = Bi. Define the evaluation map $Xt : \Omega \to R, \omega \to \omega(t)$ (21.27) that is, the restriction of the canonical projection $R[0,\infty)$; $\rightarrow R$ to ; For f, $q \in C[0,\infty)$ and $n \in N$, let dn (f, q) := ; (f - q) Ω . Further, define Z*R = ∞ N=1 N An : A1, . Monotonicity is implied by convexity. Theorem 18.8 Let X be an arbitrary aperiodic and irreducible random walk on Zd with transition matrix p. (21.46) Hence Y is a martingale and the first centered moments are Ex [(Yt - x)2] = 2x t, Ex [(Yt - x)3] = 6x t 2, Ex [(Yt - x)4] = 24x t 3 + 12x 2 t 2, (21.47) Ex [(Yt - x)6] = 720x t 5 + 1080x 2 t 4 + 120x 3 t 3. Show that v-lim $\mu n = 0$ but $n \rightarrow \infty$ that (μn) $n \in \mathbb{N}$ does not converge weakly. That is, Xn = n Rk, where R1, R2, Define Yn := k cited to the first centered moments are Ex [(Yt - x)6] = 720x t 5 + 1080x 2 t 4 + 120x 3 t 3. Show that v-lim $\mu n = 0$ but $n \rightarrow \infty$ that (μn) $n \in \mathbb{N}$ does not converge weakly. That is, Xn = n Rk, where R1, R2, Define Yn := k cited to the first centered moments are Ex [(Yt - x)6] = 720x t 5 + 1080x 2 t 4 + 120x 3 t 3. Show that v-lim $\mu n = 0$ but $n \rightarrow \infty$ that (μn) $n \in \mathbb{N}$ does not converge weakly. That is, Xn = n Rk, where R1, R2, Define Yn := k cited to the first centered moments are Ex [(Yt - x)6] = 720x t 5 + 1080x 2 t 4 + 120x 3 t 3. Show that v-lim $\mu n = 0$ but $n \rightarrow \infty$ that (μn) $n \in \mathbb{N}$ does not converge weakly. That is, Xn = n Rk, where R1, R2, R2, R3 = 120x t 4 + 120x 3 t 3. Show that v-lim $\mu n = 0$ but $n \rightarrow \infty$ that (μn) $n \in \mathbb{N}$ does not converge weakly. That is, Xn = n Rk, where R1, R2, R3 = 120x t 4 + 120x 3 t 3. Show that v-lim $\mu n = 0$ but $n \rightarrow \infty$ that ($\mu n = 0$ Xn-i. Give an example of a nonnegative martingale X with E[Xn] = 1 for all $n \in N$ but such that $n \rightarrow \infty Xn \rightarrow 0$ almost surely. In this chapter, we establish a similar stability property for martingales that are stopped at a random time. ε for all $m \ge n + 1 \cap B$. Show that the entropy H (p) is minimal (in fact, zero) if $p = \delta \varepsilon$ for some $\varepsilon \in E$. 6.2 Uniform Integrability 155 Proof " \leftarrow " Assume there is an H with the advertised properties. $x = -\infty$ Now what about symmetric simple random walk in dimensions? , ωn]. A $\varepsilon B(R)$ is a μ -null set, then dx Exercise 13.1.7 (Fundamental theorem of calculus) (Compare [37].) Let $f \in L1$ (Rd), $\mu = f \lambda d$ and let $C \subset Rd$ be open, convex and bounded with $0 \in C$., the kinetic function $E[X] : B(E) \rightarrow [0, \infty]$, $A \rightarrow E[X(A)]$ is a measure. Exercise 17.2.1 (Discrete martingale problem) Let $E \subset R$ be countable and let X be a Markov chain on E with transition matrix p and with the property that, for any x, there are at most three choices for the next step; that is, there exists a set $Ax \subset E$ of cardinality 3 with p(x, y) = 0 for all $y \in E \setminus Ax$. Definition 14.16 Let (Ω, A, μ) a measure space and (Ω', A) a measurable space. i=1 For $j \in \{\sigma k, ... That is, we (\Omega', A) \in U_{i} \inU_{i} \in U_{i} \in U_{i} \inU_{i} \inU_{i}$ consider a two-stage experiment: At the first stage we choose a realization of i.i.d. random variables $W = (Wi -)i \in Z$ on (0, 1) and let Wi + := 1 - Wi - .m = 1 n=m m $\rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$), by upper continuity of P P + $\infty + ... \in N \rightarrow \infty$ n=m + owever, for every m $\in N$ (since log $(1 - x) \leq -x$ for $x \in [0, 1]$). $\log 1 - P[An] \le \exp - P[An] = 0$. Theorem 5.3 (Rules for expectations) Let X, Y, Xn, Zn, n \in N, be real integrable random
Walks . 2 Reff (0 \leftrightarrow 1) (19.15) In particular, in the case R (0, 1) = ∞ (or equivalently Reff (0 \leftrightarrow x) + Reff (x \leftrightarrow 1)), we have Reff (0 \leftrightarrow x) = R (0, x) and Reff $(1 \leftrightarrow x) = R$ (1, x), hence $u(x) = Reff (0 \leftrightarrow x)$. For a1 < b1 < n (ai, bi], define a2 < b2 < . Theorem 11.4 (Martingale convergence theorem) Let $(Xn)n \in N0$ be a submartingale with sup{E[Xn+]: n \ge 0} < \infty. Under some mild conditions on the continuity of φ , the main contribution to the integral comes from those points x that are not too unlikely (for $\mu \epsilon$) and for which at the same time $\varphi(x)$ is large. n=1 Hence I : L2 ([0, 1]) \rightarrow L2 (P), f \rightarrow I (f) is an isometry. Corollary 19.16 Let X be a Markov chain on E with edge weights C. In some of the examples, the elements of the generating class are simpler sets such as rectangles or compact sets. For n \in N0 and t \geq 0, define the probability fn (t) := P1 [Xt > n]. \bullet $n \rightarrow \infty$ Exercise 13.2.14 Let μ , $\mu 1$, $\mu 2$, . To this end, we first study more general set functions that assign nonnegative numbers to subsets. ; ; 3 Note that ; f ; p and f d do not depend on the choice of the representative f \in f. $n \rightarrow \infty$ Hence the characteristic function can be expanded about any point $t \in \mathbb{R}$ in a power series with radius of convergence at least $1/(3\alpha)$. nn This representation of the Brownian motions goes back to Paley and Wiener who also show that along a suitable subsequence the series converges uniformly almost surely and hence the limit X is indeed continuous, see [125, Theorem XLIII, page 148]. E[$\phi(Xt) +] = E \phi E[Xt * Ft]$ (iii) This is evident since $x \rightarrow |x|p$ is convex. The remainder of this section is devoted to the proof of this theorem. Using the binomial theorem, we get (note that the mixed terms) * 1 + 2 + 3n(n - 1) E (Y1Kn) + 3n(n - 1) G 4 + 3n(n - 1) E (Y1Kn) + 2 n/2; hence we have H (x)/x $\uparrow \infty$. Example 6.29 (Laplace transform) Let X be a nonnegative random variable on (Ω , A, P). In this chapter, we define the integral by an approximation scheme with simple functions. (2.15) We need a 0-1 law similar to that of Kolmogorov. By Definition 4.13, a measurable function f : $\Omega \rightarrow [0, \infty)$ is called a density of ν with respect to μ if $\nu(A) := f 1A d\mu$ for all $A \in A$., BtN - BtN-1)]. Indeed, $\phi \alpha, \gamma = \phi \alpha, \gamma$. We now compute ∞ GY := $\infty \propto P0$ [Yt = 0] dt = 0 0 = ∞ n=0*) P0 X2n = 0, Tt = 2n dt n=0 p2n (0, 0) ∞ e-t 0 t 2n dt = G(0, 0). We formalize the description given above. $\omega k] \cap Bn = \emptyset$ for all $k, n \in N$. If we place n indistinguishable particles independently according to λ on the random positions z1, . Hence, let $\nu 0 \mu$. Hence, for efficiency, those symbols that appear more often get a shorter code than the more rare symbols. Assume that $k \in N$ and let $\phi : E k \rightarrow R$ be measurable with $E[|\phi(X 1, . Then \lambda can be extended uniquely to a measure <math>\lambda *$ on $B * (Rn) = \sigma B(Rn) \cup N$, where N is the class of subsets of Lebesgue-Borel null sets. (16.31) A function $H: (0, \infty) \rightarrow (0, \infty)$ is called slowly varying at ∞ if lim $x \rightarrow \infty$ $H(\gamma x) = 1$ H(x) for all $\gamma > 0$. (21.5) on the dyadic rational numbers and then to The idea is first to construct X extend it continuously to [0, 1]. en-1 $= P \{e0 \} \times .n \in Z$ be the canonical process on $\Omega = E Z$. Remark 6.3 Let A1 , A2 , $.1 \{x(j = 1)\}$ -i1{x(j)=x(i)} - 12, and this expression is easy to compute as it depends only on the 2d neighboring spins and, in particular, does not require knowledge of the value of Zβ. Let π be the invariant ~ Then, clearly, the product measure π ⊗π ∈ M1 (E × E) is an (and distribution of X. (b) 3 αf dµ = α f3 dµ)=X(1)} -1 Hence $\Pi(x \perp)/\Pi(x) = exp - 2b$ for $\alpha \ge 0$. (P3) If $J \subseteq I$ with $I \cap J = \emptyset$ for all I, $J \in J$ with I = J, then (NJ, $J \in J$) is an independent family. Using the generalized binomial theorem (see Lemma 3.5), we get (since we have (1 - 4p(1 - p))1/2 = |2p - 1|, if p = 12, ∞ , if p = 12, if p = 12, if p = 12, if p =measurable and $|f| d\mu < \infty$. Hence it can be developed in a power series about 0 with radius of convergence at least 1: $f(x) = \infty f(k)(0) k=0 k! xk$ for |x| < 1. + bk = n and bi \leq Bi for all i) is given by the generalized hypergeometric distribution Bk B1 ··· b1 bk. Theorem 17.15 (Reflection principle) Let Y1, Y2, . 2 f (Xn) - f (Xn-1) = 10.2 we have version X of X. This implies $E[Z1 \ E] = E[Z1]$. Let $\theta := -Cov[X, Y Var[Y]$. Then n! 12.2 Backwards Martingales 265 $E[\phi(X) \ E] = E[\phi(X) \ T] = \lim An (\phi) n \rightarrow \infty$ a.s. and in L1. This connection • in some cases allows us to distinguish between recurrence and transience by means of easily computable quantities, and • in other cases provides a comparison criterion that says that if a random walk on any connected subgraph is recurrent. For two square summable sequences (an)n \in N and (bn)n (bn)n (n)n (almost everywhere f > 0, then $\mu(A) = A f d\lambda > 0$ if $\lambda(A) > 0$; hence $\mu \approx \lambda$. } in such a way that Y is a random variables) The family (X i) i \in I of random variables) The family (σ (Xi i) particular, $G(x, y) = Px [X\tau A = y]$ for $x \in E \setminus A$ and $y \in A$. "(ii) \Rightarrow (i)" This follows from Lemma 1.31(iv). We start by recalling the Arzelà-Ascoli characterization of relatively compact sets in $C([0, \infty))$ (see, e.g., [37, Theorem 11.3]). = pe e \in E If q = p, then there is an $e \in E$ with pe > 0 and qe = pe. The supplement is trivial. Let $X^$ be a Markov chain with transition matrix $p(x, v) = 12 p(x, y) + 12 1\{x\}$ (y). Applying (7.12) with $g \equiv 1 \in L^{\infty}(\mu)$ yields $\kappa(f) \approx 1 \in L^{\infty}(\mu)$ we have 1 |Xk |p. Somewhat * more formally, we could write: If x ∈ Ei for some i, then Px Xn ∈ Ei+n (mod d) = 1. 147 147 153 160 7 Lp -Spaces and the Radon-Nikodym Theorem . (ii) Show that dH is complete if (E, d) is complete if with a counterexample for the case $\mu(\Omega) = \infty$? Any of the questions raised above can be answered using this comparison technique. (ii) If X, $Y \in L2$ (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY $\in L2$ (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY $\in L2$ (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY \in L2 (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY $\in L2$ (P), then XY \in L2 (P), then XY (X) = L2 (P), Theorems everywhere. A Exercise 15.2.2 Show that there are real random variables X, X and Y, Y with the D D D properties (i) X = X and Y = Y, (ii) X and Y are not independent. (17.13) y=x Then we define q(x, x) = -q(x, y). This is the kernel on $[0, \infty)$ whose Laplace transform is given by ∞ $\kappa t(x, dy) = -\lambda y = \psi t(\lambda)x$. It is enough to consider the case $f \ge 0$. If κ^2 is a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes
\kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product $\kappa 1 \otimes \kappa^2$ similarly by formally understanding κ^2 as a kernel from ($\Omega 1$, A1) to ($\Omega 2$, A2), then we define the product κ^2 similarly by formally understanding κ^2 sis a kernel formal (\kappa^2) = 1 sis a divisible. By $\varphi(t) = x \in ZD$ ei)t, x* p(0, x) denote the characteristic function of a single transition. In particular, the paths are almost surely nowhere differentiable. Theorem 19.22 (Thomson's (or Dirichlet's) principle of minimization of energy dissipation) Let I and J be unit flows from A1 to A0 (that is, I (A1) = J (1). Definition 7.35 Let μ and ν be two measures on (Ω, A) . Use Exercise 13.1.5 to show that $\lim r \downarrow 0$ $\mu(x + rC) = 0$ rd for λd -almost all $x \in A$, k, we have *) P Xir \in Air for all r = 1. In this case, we have ' ((' p n (x, y)p(y, x) = Px Xn = y; $\tau x 1 > n$; $Xn + 1 = x = Px \tau x 1 = n + 1$. In fact, in many cases, it is enough to consider moments, Laplace transforms or characteristic functions., Dk,dk \in A i=1 Ck,i = Bk \subset A k such that Ak \ Bk = di=1 Dk,i . (ii) A map F : E N \rightarrow E is called a current flow I : E \times E \rightarrow R on E \ A is called a current flow if there exists a function u : E \rightarrow R with respect to which Ohm's rule is fulfilled: I (x, y) = u(x) - u(y) R(x, y) for all x, y \in E, x = y. , k + m = m + n - m 1Am $\circ \tau k$., k + 1) $\subset \sigma (Xm, l, l \in N)$. Let E = N and let X = (Xt) t ≥ 0 be a Markov process on E with Q-matrix $[| x, q(x, y) = -x, | [0, if y = x + 1, if y = x, else. For any finite I = {i1, . If we had <math>\mu = 0$, then there would exist (since $\mu(E) = 0$) points x1, x2 $\in E$ with $\mu(\{x1\}) > 0$ and $\mu(\{x2\}) < 0$. Clearly, for every $y \in E$, (x1, y) + this would imply $\mu(\{x1\}) p = \mu(\{x2\}) p(x2, y) = \mu(\{x1\}) p(x1, y) + \mu(\{x2\}) p(x2, y) = \mu(\{x1\}) p(x1, y) =$ should like to thank all those who read the manuscript and the German original version of this book and gave numerous hints for improvements: Roland Alkemper, René Billing, Dirk Brüggemann, Anne Eisenbürger, Patrick Jahn, Arnulf Jentzen, Ortwin Lorenz, L. In both cases, let $\nu * 0 := \delta 0$. s>0 Then P[A] $\in \{0, 1\}$. Definition 17.12 Let I $\subset [0, \infty)$ because of the manuscript and the German original version of this book and gave numerous hints for improvements: Roland Alkemper, René Billing, Dirk Brüggemann, Anne Eisenbürger, Patrick Jahn, Arnulf Jentzen, Ortwin Lorenz, L. In both cases, let $\nu * 0 := \delta 0$. s>0 Then P[A] $\in \{0, 1\}$. Definition 17.12 Let I $\subset [0, \infty)$ because of the manuscript and the German original version of this book and gave numerous hints for improvements: Roland Alkemper, René Billing, Dirk Brüggemann, Anne Eisenbürger, Patrick Jahn, Arnulf Jentzen, Ortwin Lorenz, L. In both cases, let $\nu * 0 := \delta 0$. closed under addition. Now use the fact that $\pi pn = \pi$ and $\mu x pn = \mu x$ for all $n \in N$ to conclude that even $\pi(\{y\}) = \pi(\{x\})\mu x$ ($\{y\}) = \pi(\{x\})\mu x$ ($\{x\})\mu x$ ($\{x\}, \mu x$) ($\{x\}, \mu x$ C B E[$\xi k \xi l$] 1[0,s], bk 1[0,t], bl k,l=1 = n B CB C 1[0,s], bk k=1 C B $\rightarrow \rightarrow 1[0,s]$, 1[0,t] = min(s, t). Then Xn (ω) = X0 (τ n (ω)). A A Proof Let X : A $\rightarrow \Omega$, $\omega \rightarrow \omega$ be the canonical inclusion; hence X-1 (B) = A $\cap B$ for all B $\subset \Omega$. Firstly, one can solve the system of linear equations belonging to the Markov chain killed in the two points. Assume that f is convex on $[0, \infty)$. Sn = E[X1] > 0 almost surely. Clearly, we have pn0 (x0, y) = (pA)n0 (x0, y) for all $y \in A$. Sup Xn+1 - Xn < ∞ ., Sn }, n \in N. Example 17.18 (Random walk on Z) Let E = Z, and assume p(x, y) = p(0, y). -x) for all x, y $\in \mathbb{Z}$. (2.7) $j \in J$ Example 2.12 As in Example 2.4, let (Ω, A, P) be the product space of infinitely many repetitions of a random experiment whose possible outcomes e are the elements of the finite set E and have probabilities $p = (pe)e \in E$. k=1 We show that $\mu((a, b)) \leq \infty$ $\mu(a(k), b(k)]$. Therefore, lim sup $\mu(E \setminus K) \leq \lim_{n \to \infty} \sup_{n \to \infty} \mu(E) = 0$ $\rho L, K d\mu n \rightarrow \infty n \rightarrow \infty = \mu(E) - \rho L, K d\mu \leq \mu(E \setminus L) < \epsilon. 45$ For $N \in N$, the random variables $XN := -N 4 N 5 2 2 |Y| \land N$ take only finitely many values and are independent as well. \blacklozenge Definition 7.40 (Signed measure) A set function $\phi : A \rightarrow R$ is called a signed measure on (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is σ -additive; that is, if for any sequence of (Ω, A) if it is $(A \cap A)$ if it pairwise disjoint sets A1, A2, Clearly, $(\Omega, A, P, \tau r)$ is a measure-preserving dynamical system. Show that convergence in probability implies almost everywhere convergence. For every $\alpha > 0$, $1 \nu(\{|f| > \alpha\}) \leq F 1 \cdot 1\{|f| > \alpha\}$. Then the following statements hold. Let $C \subset Cb$ (E; K) be a family that separates points; that is, stable under multiplication and that contains 1. By approximating f by simple functions, we see that the right-hand side in (8.18) is F -measurable (see Lemma 14.23 for a formal argument). .) \subset En for $n \in N$; hence $T \subset E$. Em B \in Pn+ $n \setminus E n-1$ B \subset Em m $n \in E n-1$ B \subset Em m $n \in$ variables with 1 lim sup $E[|X|n] 1/n < \infty$, $n \to \infty$ n 1 lim sup $E[|Y|n] 1/n < \infty$, $n \to \infty$ n 354 15 Characteristic Functions and the Central Limit Theorem and E[Xm Y n] = E[Xm] E[Yn] for all m, $n \in \mathbb{N}0$. If the offspring number has a second moment, then Zn grows of order mn and Zn/mn is uniformly integrable. The essential assumptions are that the summands are independent, each summand contributes only a little to the sum and the sum an -algebra F. Denote by $\Omega = \times \Omega$ is the set of maps $\omega : I \rightarrow \Omega$ is such that $\omega(i) \in \Omega$ is left or all $i \in I$. Thus we define a new probability measure PB on (B, 2B) by PB [C] = #C #B for C \subset B. Now restrict ϕ to that flat piece and inductively reduce its dimension until reaching a point, the case that has already been treated above. However, here we only sketch the proof since we do not want to go into the details of signed measures and signed contents. For example, VT could be a (European) call option with maturity T and strike price $K \ge 0$. Finally, we present a third method of studying recurrence and transience of random walks that does not rely on the Euclidean properties of the integer lattice but rather on the Fourier inversion formula. We introduce the notion of weak convergence of probability measures on general (mostly Polish) spaces and derive the
fundamental properties. , Xn) such that ξn (X) = ν : n 1 An (ν) := k = (k1, . If the X1, X2, . k $\rightarrow \infty$ Furthermore, for open A and for C \in C with C \subset A, α (C) = lim µnk (C) \leq lim inf µnk (A), E I . = 1 Exercise 19.5.4 For the graph of Fig. For irreducible chains all states are in the same class: either positive recurrent, null recurrent or transient. : $\Omega \rightarrow E$ be measurable with respect to A - B(E). For convenience, also the diagonal is drawn. k=1 In the case d = 1, the following theorem goes back to Bochner (1932). Note that the limit does not depend on the choice of the subsequence and is thus unique. $y = \phi - 1$ (x) $-\phi - 1$ (y) for x, $y \in \mathbb{R}$ defines a metric on \mathbb{R} such that In fact, $d(x, \phi \text{ and } \phi - 1 \text{ are continuous}$. Klenke, Probability Theory, Universitext, 515 516 21 Brownian Motion (ii) indistinguishable if there exists an $\mathbb{N} \in \mathbb{R}$ with $\mathbb{P}[\mathbb{N}] = 0$ such that $\{Xt = Yt\} \subset \mathbb{N}$ for all $t \in \mathbb{I}$. If an \downarrow inf f (K) is strictly monotone decreasing, then $K \cap f - 1$ ($[-\infty, an]$) = \emptyset is compact for every $n \in N$ and hence the infinite intersection also is nonempty: f - 1 ($[-\infty, an]$) = \emptyset . Let Sn = n - 1 Xk k=0 denote the nth partial sum. 16.1 Lévy-Khinchin Formula 375 Example 16.15 For an infinitely divisible distribution μ on $[0, \infty)$, we can compute the Lévy measure ν by the vague limit $\nu = \nu$ -lim $n\mu * 1/n \quad n \to \infty$ (0, ∞) (16.7). In the reduced network, the effective resistances are easy to compute: If {a, b, c} = {0, 1, x}, then Reff (a \leftrightarrow b) = 1 1 + R (a, b) R (a, c) + R (b, c) -1. Hence 1 \cap { $\tau \leq n$ }. Show that f is (properly) Riemann integrable if and only if f is λ -a.e. continuous. Denote by Pn the partition that is generated by the sets $n-1 = 0 \tau$ (Ail), i1 , . We say that X explodes. Since $\mu 1$ and $\mu 2$ are locally finite, for every $x \in K$, there exists an open set Ux x with $\mu 1$ (Ux) < ∞ and $\mu 2$ (Ux) < ∞ . In other words, (X1 , . As uniform limits of continuous functions are continuous. Loosely speaking, at the boundaries of closed sets, mass can immigrate but not emigrate. This implies that $E \in DB$ for any $B \in \delta(E)$. 17.3 Discrete Markov Processes in Continuous Time . Proof Clearly, 0 and Y = X - a are submartingales. (p) (1 - p) i i=A If we define p = (1 + b)p*, then $p \in (0, 1)$ and 1 - p = (1 - p*)(1 + a). 5.2 Rolling a die n times: For a fixed realisation, the values of Sn /n converge to 3.5. We have n on the horizontal axis., Xn (ω)?, E12 are countable. Then Z := X Λ Y = (min(Xt, Yt))t \in I is a supermartingale. Since μ is upper semicontinuous (Theorem 1.36), there is a $\delta > 0$ such that $\mu(B\delta) \leq \mu(B) + \epsilon$. We then introduce martingales and the discrete stochastic integral. k=1) * Therefore, Ex $\tau x1 = 1$ $\pi(x) < \infty$, and thus X is positive recurrent. \bullet Proof (ii) As μ and the outer measure μ * coincide on σ (A) and since $\mu(A)$ is finite, by the very definition of μ * (see Lemma 1.47) there exists a covering B1, B2, . Here also the convergence has exponential speed and the rate is determined by the second largest eigenvalue of p. + - For any 3 measurable $3map f: \Omega \to R$, we have $3 f \le |f|$ and $f \le 3|f|$, - which ± implies $3 f d\mu \le |f| d\mu$. $n \to \infty$ By Lévy's continuity theorem, as a continuous limit of CFPs, ϕ is a CFP. Hence convergence in L1. By the triangle inequality, $d(x, z) + d(z, y) \ge d(x, y)$ for all $x, y \in E$ and $z \in F$. Further, we agree on the following notation for spaces of continuous functions: $C(E) := f \in C(E)$ is bounded, $Cc(E) := f \in C(E)$ is bounded, $Cc(E) := f \in C(E)$ is bounded, $Cc(E) := f \in C(E)$ has compact support $\subset Cb(E)$. For the first equivalence, we distinguish two cases. Hence, in this case, the speed of convergence is known precisely. The following proof shows that this formal argument can t =0 be made rigorous. Evidently, gn(x) = gm(x) for every $x \le n$ for every $x \ge n$ fo Theorem 6.25, we thus have lim E[Yn] = 0. Without loss of generality, assume $f \propto \in (0, \infty)$. Now let (fn)n \in N be a sequence of simple functions with fn $\uparrow f$., Cn \in A such that B \ A = ni=1 Ci. In particular, if the network is irreducible, an electrical potential is uniquely determined by the values on A. Due to the monotonicity, we can make the following definition. A complete normed vector space is called a Banach space. Let X be as above and let Z be a σ (X)-measurable real random variable. Then, for any i = 1, . Evidently, the distribution function of (X1, X2) is F., L} the set of those edges with both vertices lying in BL. Then u is also 2n-times differentiable at 0 and u(2k-1)(0) = 0 for k = 1, Let $F = \sigma(X)$ be the filtration generated by X and define $F := + + (Ft)t \ge 0$ by Ft = s > t Fs. Rather, we have to peel off the negative portions layer by layer. 22.2 Skorohod's Embedding Theorem . Then the following statements hold: (i) A is \cap -closed. 5.5) has the Q-matrix $q(x, y) = \alpha(1\{y=x+1\} - 1)$ $1{y=x}$). Define the left continuous inverse of F : F -1 (t) := inf{x \in R : F(x) \ge t} for t \in (0, 1). = Poiu1 (A) * Poiu2 (A) * . \ddagger 15.2 Characteristic Functions: Examples Recall that Re(z) is the real part of $z \in C$. 458 18 Convergence of Markov Chains An alternative approach to the eigenvalues can be made via the roots of the characteristic polynomial $\chi N(x) = det(p - xI)$, $x \in R$. The map $\nu \to m(\nu)$ is continuous; hence EA is open (respectively closed) if A is open (respectively closed) if A is open (respectively closed). (French: portemanteau). Here we give a construction for X that could actually be used to implement a computer simulation of X. 15.5 The Central Limit Theorem . • 8.3 Regular Conditional Distribution 205 Definition 8.28 Let Y be a random variable with values in a measurable space (E, E) and let F C A be a sub- σ -algebra. Proof (i) Let x \in I \circ . 25.4 Dirichlet Problem and Brownian Motion .. Then) * 1 $\phi(t)$:= E et X1 = R et x et x $\mu(dx)$ = 1 $\phi(t + \tau)$. In addition, we assume that there is an exterior magnetic field of strength h. A We will show f \leq 0. Ceff (A0 \leftrightarrow A1). However, if we choose c = 0, then Sk2 1A⁻ k \geq t 2 1A⁻ k . 19.13). α Remark 15.27 In fact, $\phi\alpha$, r is a characteristic function for every $\alpha \in (0, 2]$ ($\alpha = 2$ corresponds to the normal distribution), see Sect. \circ Fn (x) = Fn \circ . By the Kolmogorov-Chentsov theorem (Theorem 21.6(ii)), for $\epsilon > 0$ and $\gamma \in (0, \beta/\alpha)$, there exists a K such that, for every $i \in I$, we have) * P |Xti - Xsi | $\leq K |t - s|\gamma$ for all s, $t \in [0, N] \geq 1 - \epsilon$. The transition probabilities are given by stochastic matrices. On An, we have $S_{Tn} + 1 < L - \varepsilon$. Takeaways Almost everywhere (almost sure) convergence implies stochastic convergence implies stochastic convergence. \blacklozenge C(y) (19.7) Definition 19.11 Let (E, K), C and X be as in Example 19.10., Xn) is independent and hence, by Theorem 2.13(ii), (Xn)n \in N is independent as well. For t \in [0, 1] and n \in N0, define t Bn (t) = bn

(s) $\lambda(ds)$; 0 540 21 Brownian Motion that is, B0 (t) = t and \sqrt{Bn} (t) = 2 sin(n \pi t) n \pi for $n \in N$. 7.4 Lebesgue's Decomposition Theorem . Example 23.15 Let $\Sigma \subset Rd$ be finite and let μ be a probability measure on Σ . 1 + R 2 + R 3 R x 3 R 2 R 3 R 1 x 1 z R 1 x 1 R 2 x 2 Fig. \blacklozenge Corollary 9.34 Let X be a submartingale and $a \in R$. Hint: Proceed as in the proof of Lévy's continuity theorem. Since we modeled the clicks as a Poisson process with intensity α , this probability can easily be computed:) * P N(s,s+t] = 0 = e-\alpha t. 24.1 Random Measures . Further, let ϕ be the characteristic function of (X(A1), . Now let P[X1 < 0] > 0. \blacklozenge We now come to the situation of the general dynamical system (iv) Let μ be a distribution on Rn and let X be a random variable with PX = μ . As a further application, we get the 0-1 law of Hewitt and Savage [72]. (For example, consider $\Omega = \Omega = R$, A = B(R), and $X(\omega) = \omega$ for all $\omega \in \Omega$. (ii) Use Exercise 15.3.2 to infer that $\mu = \delta 0$ if $\alpha > 2$. Then, for any $t > s \ge 0$, there is an $N \in N$ with TN > t. Mainz October 2007 Achim Klenke Contents 1 Basic Measure Theory . The chapter finishes with the investigation of random stopping times with an infinite time horizon. 14.2 Finite Products and Transition Kernels 311 For two finite measures $\mu, \nu \in Mf(Rn)$, define the convolution $\mu * \nu \in Mf(Rn)$, define the convolution $\mu * \nu \in Mf(Rn)$, define the convolution $\mu * \nu \in Mf(Rn)$, define the convolution $\mu * \nu \in Mf(Rn)$, define the convolution $\mu * \nu \in Mf(Rn)$ by $(\mu * \nu)((-\infty, x]) = 1Ax (u, v) \mu(du) \nu(dv)$, where $Ax := \{(u, v) \in Rn \times In (u, v) \in Rn \times$ Rn : $u + v \le x$. The (probability) generating function (p.g.f.) of PX (or, loosely speaking, of X) is the map ψ PX = ψ X defined by (with the understanding that 00 = 1) ψ X : $[0, 1] \rightarrow [0, 1]$, $z \rightarrow \infty$ P[X = n] zn . Proof (i) Let X = M + A be Doob's decomposition of X. If B(Rn) were a topology, then it would be closed under arbitrary unions. In order to show that $\delta(E)$ is a π -system, it is enough to show that $\delta(E) \subset DB$ for any $B \in \delta(E)$. (i) τC is an F-stopping time (and an F+ -stopping time). \blacklozenge Reflection Find an example of an exchangeable family (Xn) $n \in N$ of $\{0, 1\}$ -valued random variables that is not independent. \blacklozenge Let X = (Xn) $n \in N$ be a stochastic process with values in a Polish space E. f (ϵ) \clubsuit Exercise 8.2.6 Show the conditional Cauchy-Schwarz inequality: For square integrable random variables X, Y, $E[XY | F] \ge E[X 2 | F]$. Hence the limit t (ω) = enH (ν) R Step 4. Theorem 18.4 Let X be irreducible with period d., we obtain $[\omega 1, .] \in [150]$ and [151]). Since $\#J^{\sim} = n + 1$, this verifies the induction step. Then, for any $c \in R$, we have $f - 1((c, \infty)) \in [0, 1]$. P[A] P[A] Now use the expression in (8.2) for P[A]. Gibbs Sampler We consider a situation where, as in the above example, a state consists of many components $x = (x_i) \in A \in E$ and where A is a finite set. n=1 Remark 1.32 The inequality in (iv) can be strict (see Example 1.30(iii)). Somewhat more generally, an undirected graph G is a pair G = (V, E), where V is a set (the set of "vertices" or nodes) and $E \subset \{\{x, y\} : x, y \in V, x = y\}$ is a subset of the set of subsets of V of cardinality two (the set of subsets i,σ for some $i \in \Lambda$, else, is called a Gibbs sampler for the invariant distribution π . In contrast to the case Mf (E), the function 1 is not integrable. Thus (2.3) does not hold and so the events A1 , A2 , A3 are not integrable. Thus (2.3) does not hold and so the events A1 , A2 , A3 are not integrable. 0 if N0 = 0 and if: (i) For any $n \in N$ and any choice of n + 1 numbers 0 = t0 < t1 < . We come back to this point in Chap. Based on this, construct a set that is not Borel and whose closure is a null set.) & Exercise 4.3.4 Let $f: [0, 1] \rightarrow (0, \infty)$ be Riemann integrable. In this case, integrable if and only if $\omega \in \Omega$ $f d\mu = f(\omega) \alpha \omega$. Pólya's theorem gives a sufficient condition for a symmetric real function. Further, let $w : E \rightarrow R$ be a function that is constant both on A0 and $w \equiv : w0$ and $w \equiv : w0$ and w = : w0 and w = : w0 and w = : w1. Assume that for all $n \in N0$, given X0, . By the conditional version of the Borel-Cantelli Lemma (see Exercise 11.2.7), we infer ('P lim sup An = 1. be independent random variables with Xi ~ Berp, $i \in N$., $An \in E$ with nk=1 Ak = A and $\mu(Ak) = \mu(A)/n$ for any k = 1, . If F is not a constant map, then go to (2). Furthermore, $/i \in I$ Ai = $\sigma(Z)$. Hint: Use induction on n. We write $\mu 1 \leq st \mu 2$ if $f d\mu 2 \leq st \mu 2$ if f $(x) \in Ex$ and thus $\Lambda * (x) = 1 * (x)$, $x^* - \Lambda(t * (x)) = inf H (\nu | \mu) = I^{(x)}$. Define $h = \delta(\varepsilon) h$. $(y, x) F = 1 | \tau x 1 - 1 n = \tau y$ and Y be independent Poisson random variables with $(x, y) = Px Xn = y; \tau x 1 > n$. By Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 68 2 Independence Example 6.29, we can compute the moments of Zn by differentiating the Laplace transform. 70 and 10 an parameters μ and $\lambda \ge 0$. Let $i \in \{0, . Note that A1 = A2 = A3 = . A postulate of statistical physics is that the distribution: <math>\mu\beta n$ (dx) = (Zn β) – 1 e- β Un (x) μ 0n (dx). for all $A \subset N0$. Then P is called the Wiener measure. In a manner similar to the above, we make the following definition. it follows that $\mu E \setminus A \le 0$. $E \setminus A < n=1$ Theorem 13.6 If E is Polish and if $\mu \in Mf(E)$, then μ is regular. 19.3 Finite Electrical Networks .. Exercise 20.3.1 Let (Ω, A) be a measurable map. , Xn be independent random variables with values in 1, . In order to check it on a generator of the σ -algebra (Theorem 2.16). Here we follow the alternative route as described in [13] (and later [14]) or [44]. The $l \rightarrow \infty$ 3 3 3 map
$\xi \rightarrow F$ d $\xi = f1$ d $\xi \cdots fk$ d ξ is bounded and (as a product of continuous with respect to the topology of weak convergence on M1 (E); hence it is in Cb (M1 (E)). The random variable S := ∞ n=1 Xn is measurable with respect to E but not with respect to T. This can be employed to prove the general case., kD, kD (17.20) is the multinomial coefficient. If by some clever choice of the distribution of Fn one can ensure that the stopping time T := inf{n \in N : F1n is constant} is almost surely finite (and this is always possible), then we will have P[F1T (x) = y] = $\pi(y)$ for all x, $y \in E$., then this is exactly Bayes' formula of Theorem 8.7. Sec. ($\lambda(\phi \nu(t) - 1)$) is the CFP of $\mu \lambda = \infty k = 0 \text{ e } k!$ write $\mu \lambda = e * \lambda(\nu - \delta 0)$. Then μ is uniquely determined by the values $\mu(E)$, $E \in E$. ($\lambda(\phi \nu(t) - 1)$) is the CFP of $\mu \lambda = \infty k = 0 \text{ e } k!$ write $\mu \lambda = e * \lambda(\nu - \delta 0)$. Then μ is uniquely determined by the values $\mu(E)$, $E \in E$. ($\lambda(\phi \nu(t) - 1)$) is the CFP of $\mu \lambda = \infty k = 0 \text{ e } k!$ write $\mu \lambda = e * \lambda(\nu - \delta 0)$. Then μ is uniquely determined by the values $\mu(E)$, $E \in E$. ($\lambda(\phi \nu(t) - 1)$) is the CFP of $\mu \lambda = \infty k = 0 \text{ e } k!$ write $\mu \lambda = e * \lambda(\nu - \delta 0)$. Exercise 13.4.1 Show that a subset $K \subset M1$ (M1 (E)) is tight if and only if, for any $\varepsilon > 0$, there exists a compact set $K \subset E$ with the property $\mu \mu \in M1$ (E) := $\kappa(Xs, \cdot)$. Exercise 17.4.1 Let x be positive recurrent and let F(x, y) > 0. We call $H(p) := \kappa(Xs, \cdot)$. He (p) (e = 2.71. Clearly, that condition is not necessary, as, for example, the normal distribution does not fulfill it. k=0 Proof First we show that, for fixed $x \in E$, (14.16) defines a probability measure $n-1 / \kappa_j k$, jk+1. In order that the random walk be at the origin after 2n steps, it must perform ki steps in the opposite direction for some numbers k1, By the unique -; hence (see ness theorem for probability generating functions, we get $Y \sim bn, p - *n$ Definite measures on (Ω, A) with $\nu 0 \mu 0 \alpha$. Define the map $m : E \rightarrow Rd$, $\nu \rightarrow x \nu(dx) = x \nu(\{x\})$. Let (zn) $n \in N$ be a sequence in F $n \rightarrow \infty$ with $zn \rightarrow \infty$ with $zn \rightarrow x$. In particular, if $x \in E$ and $\sigma x = \inf\{n \in N0 : Xn = x\}$, then $\sigma x < \infty$ since X is recurrent and irreducible. Takeaways In order to draw random samples (approximately) according to a given distribution, it is sometimes feasible to simulate a suitable Markov chain that converges to this distribution as its invariant measure. For convenience, we recall the construction of Z. Corollary 7.44 (Jordan's decomposition theorem) Assume $\phi \in M \pm (\Omega, A)$ is a signed measure. By partial $y_0 = 1$ $y_1 \phi(t) y_2 y_3 - a_4 - a_3 - a_2 - a_1 a_1 a_2 Fig.$ (i) Give a formal description of this process as a Markov chain. 232 10 Optional Sampling Theorems The process An = # i $\leq n - 1$: |Xi | = 0 is the socalled local time of X at 0. Furthermore, for E = R and $\gamma > 1$, every locally Hölder- γ -continuous function is constant. * random variables is called symmetric if finitely many of the Xi can be permuted without changing the event. Choose an arbitrary probability vector $(gn) \in N$ with gn > 0 for all $n \in N$. Sometimes we $n \rightarrow \infty$ D $n \rightarrow \infty$ write $Xn \rightarrow PX$ if we want to specify only the distribution PX but not the random variable X. In general dynamical systems, a similar statement is true if we replace the average over ω by the conditional expectation given the σ -algebra of invariant events. $\in A$ with ∞ Ai $\in A$, i=1 (iv) subadditive if for any choice of finitely many sets A, A1, Let $r \in \{1, ..., 1445 \text{ Moments and Laws of Large Numbers Let W1, W2, ..., The above calculation with t replaced by it yields <math>\phi(t) = \theta/(\theta - it)$, and this function is indeed analytic. \blacklozenge Example 17.6 In the previous example, it is simple to pass to continuous time; that is, I = [0, ∞). 573 573 576 583 23 Large Deviations ..., kn) $\in \Sigma$ n : δ ki = ν . Let (pe) e \in E be a probability vector. Let F ∞ := σ (Fn : n \in N), and let M be the vector space of uniformly integrable F-martingales., μ n be finite measures or, more generally, Lebesgue-Stieltjes measures on R, B(R). By Exercise 5.1.3, these are the moments of the Beta distribution β M,N-M on [0, 1] with parameters (M, N - M) (see Example 1.107(ii)). (ii) If X is nonnegative and if $\tau < \infty$ a.s., then we have $E[X\tau] \le E[X\sigma] < \infty$, $E[X\sigma] < \infty$, and Bj \subset m i=1 Ai if β j = 0. We now come to a formal description of the model. i=1 λ n is called the Lebesgue measure on Rn , B(Rn) or Lebesgue measure (It can be shown that dP (μ , ν) = dP (ν , μ) if μ , $\nu \in M1$ (E).) If E is locally compact and Polish, then (Mf (E), τ w) is again Polish (see [136, page 167]). variables and let Sk = X1 + . 11.3 Example: Branching Process Let $p = (pk)k \in N$ on verges for all $q \in Q$. i=0 i=0 Corollary 14.46 (Measures by consistent families of kernels) Under the assumptions of Theorem 14.45, for every probability measure μ on E I, B(E) \otimes I with the following property: For any choice of finitely many numbers 0 = j0 < j1 < /j2 < . On the other hand, for A \in $n \in N$ Ftn and $\sigma > \tau$ a stopping time, we have for all t Ft A \cap { $\tau n \leq t$ } \cap { $\sigma \leq t$ } = A \cap {($\sigma \lor \tau n$) \leq t} $\uparrow A \cap \{\sigma \leq t\}$. \clubsuit Exercise 4.3.2 Let $f:[0, 1] \rightarrow R$ be bounded. If F is a topological space and $m: E \rightarrow F$ is continuous, then the image measures ($\mu \epsilon \circ m-1$) $\epsilon > 0$ satisfy an LDP with rate function $I^{\sim}(x) = \inf I \pmod{\varphi}$ n by pn (0, $x = (2\pi) - D \left[-\pi, \pi\right] D e^{-i}t, x^* \varphi n$ (t) dt. Note that " $\leftrightarrow \phi p$ " is an equivalence relation; however, a p random one, as it depends on the values of the random variables (Xe) $e \in E$. Hence, letting $L = \epsilon N$, we get lim inf GN (0, 0) $\geq N \rightarrow \infty 12\epsilon$ |y| $\leq \epsilon k p k (0, y) = 1$ for for every $\epsilon > 0.0$ For $\epsilon \delta \leq 3$ one can choose $C = 12/\delta 2 \epsilon 3$., n, define Mn,t,l = # s $\leq t : Xs \in A$ ((l-1)/n, l/n] and the number of nonempty boxes after t balls are thrown: Nn,t := n 1{Mn,t,l > 0}. By convexity, we have I (y) > I (x) whenever y > x ≥ 0 or y < x ≤ 0., N - 1}2). n→∞ Thus X < ∞ almost surely and α + nk=0 Xk ⇒ X. A set A ⊂ E is called dense if A = E. If K = C, then in addition assume that C is closed under complex conjugation. Irreducibility of the Gibbs sampler, however, has to be checked for each case. Then the map Lp (Ω , A, P) \rightarrow Lp (Ω , F, P), X \rightarrow E[X |F], is a contraction (that is, E[X we know the value X = x, the random variables Y1, Since d is continuous, we have $d(x, zn) + d(zn, y) \rightarrow d(x, y)$. At the first stage, we determine the value of X. Then, for any $A \in A$, P[A] = P[A|Bi] P[Bi]. Theorem 18.13 (Convergence of Markov chains) Let X be an irreducible, positive recurrent Markov chain on E with invariant distribution π . It is used to localise the convergence. Let A, $B \in E \otimes I$. For the following theorem, compare Definition 9.7. Theorem 14.50 For any convolution semigroup ($\nu t : t \in I$) and any $x \in Rd$, there exists a probability measure Px on the product space (Ω , A) = (Rd) I, B(Rd) \otimes I such that the canonical process (Xt) t $\in I$ is a stochastic process with Px [X0 = x] = 1, with stationary independent increments and with Px \circ (Xt - Xs)-1 = ν t -s for t > s. + Tn-Ln] * wLn r = P TLs n +1 < Tn-L = . (iii) If τ is an F-stopping time and X is adapted, then X τ is an F-stopping time and X is adapted, then X τ is an F-stopping time and X is adapted, then X τ is an F-stopping time and X is adapted, then X τ is an F-stopping time and X is adapted, then X τ is an F-stopping time and X is adapted. Then (as in the proof of Wald's identity) Sn and $1{T = n}$ are independent; hence Sn2 and $1{T = n}$ are uncorrelated and thus ∞ (' (' E ST2 = E 1{T = n} Sn2 n=0 = ∞ ' (E[1{T = n} Sn2 n=0 = ∞ E[X1]2. On the other hand, limes superior is the event where infinitely many of the An occur. Hence the strong law of large numbers is in force. n k=1 k=1 For n = 1, this is clear. Assume that (µn) n \in N does not converge weakly to µ. (i) Show that E[Gn(t)] = 0 and $Cov[Gn(s), Gn(t)] = s \land t - st$ for s, $t \in [0, 1]$. 435 435 439 445 453 19 Markov Chains and Electrical Networks . In statistical physics, one is often integrating with respect to $\mu\epsilon$ (where $1/\epsilon$ is interpreted as "size of the system") functions that attain their maximal values away from the zeros of I. Proof Let ϕ : $R \rightarrow [0, 1]$, $t \rightarrow (t \vee 0) \wedge 1$. Hence, using the triangle $f d\mu 1 - f d\mu 2 \leq \epsilon 2 f \infty + 2\epsilon + \mu 1$ $(\text{Rd}) + \mu 2$ (Rd). 5.1 Moments 119 (iii) Let $\mu \in \text{R}$ and $\sigma 2 > 0$, and let X be normally distributed, X ~ N $\mu,\sigma 2$. k=1 Again, by the ergodic theorem, 1 (20.6) Rn $\leq P[\text{Am I}]
\rightarrow P[\text{A}|\text{I}]$ almost surely (by Theorem 8.14(8.14)), the claim follows from (20.5) and (20.6). (v) Now let τ be an arbitrary stopping time. be i.i.d. real random variables that satisfy the condition of Cramér's theorem (Theorem 23.3); i.e., $\Lambda(t) = \log(E[et X1]) < \infty$ 596 23 Large Deviations for every $t \in \mathbb{R}$. (i) Let μ be a measure on \mathbb{R} , $B(\mathbb{R})$ with density f with respect to the 3 Lebesgue measure λ . Rd A map $f : \rightarrow \mathbb{R}$ is called partially continuous at $x = (x1, ..., \Lambda(t) = \log(E[et X1]) < \infty$ 596 23 Large Deviations for every $t \in \mathbb{R}$. obtain a Hilbert space (V0,) \cdot , \cdot *0). 592 23 Large Deviations Let X¹, X² 2, ., 2n) are independent and identically distributed. The meaning of (P5) is explained by the following calculation. To show continuity at t = 0, consider lim sup Xt = lim sup t $\rightarrow \infty$ t $\downarrow 0 \leq \lim sup n \rightarrow \infty 1$ Bt t 1 1 Bn + lim sup Sup Bt – Bn, t \in [n, n + 1]. The bounded harmonic functions are constant, but what are the unbounded harmonic functions? A Markov process (Xt) t \in I with distributions (Px, x \in E) has the strong Markov process (Xt) t \in I with distributions (Px, x \in E) has the strong Markov property if, for every a.s. finite stopping time τ , every bounded B(E) \otimes I – B(R) measurable function f : E I \rightarrow R and every x \in E, we have Ex) * f ((X τ + t) t \in I) F τ = EI κ (X τ , dy) f (y). The procedure imitates the proof that Ω is compact. If $\sigma 2 \ge 0$ and $b \in \mathbb{R}$, then ($\sigma 2$, b, ν) is called a canonical triple. If all level sets I -1 ([$-\infty, a$]), $a \in [0, \infty)$, are compact, then I is called a canonical triple. If A1, A2, . On the other hand, again by Theorem 12.17, *) $n \rightarrow \infty$ An ($\phi k - 1$) $- \rightarrow E \phi k - 1$ (X1, . Now we show that r must equal 0, which contradicts the assumption Pp [N ≥ 3] > 0. By computing the cases Xn = Xn - 1 - 1 and Xn = Xn - 1 - 1 of (Xn - 1 - 1) + f(Xn - 1) + f(Xn - 1 - 1) + f(Xn $(\nu/\nu(R))*n$ (and $\nu *n = 0$ if $\nu = 0$). In order to simulate a chain X that converges to π , we take a reference chain with transition matrix q proposes a transition from the present state x to state y, then we accept this proposal with probability $\pi(y) q(y, x) \wedge 1$. In Example 10.19 (Equation (10.7)) for the case r = 12, and Example 10.16 for the case r = 12, it was shown that, for every $\mu \in M1$ (E), $n \rightarrow \infty \mu pn \rightarrow (1 - m(\mu))\delta 0 + m(\mu)\delta 0 + m(\mu)\delta 0 + m(\mu)\delta 0 + m(\mu)\delta 0$. Then $A \in T = \infty \sigma$ (Xn, Xn+1, . Indeed, by the monotonicity principle, we have (E,K) Reff (22, L2) ($0 \leftrightarrow \infty$) = ∞ . Takeaways An event that is described by a sequence X1, X2, . We will construct a binomially distributed random variable by throwing a Poi λ -distributed number T of balls in ni boxes and count the number of nonempty boxes. Choose an open set $U \supset (A \cap [-N, N]n)c$) < $\epsilon/2$, and let $K := [-N, N]n \setminus U \subset A$. Try and fill the details in this argument. , $n \ge \# k \le n : Sl = Sk$ for all l > k= n 1A $\circ \tau k$. The supplementary statement is simple and is left as an exercise. Indeed, for the case where f (x) = ∞ for all $x \in K$, the statement is trivial. $n \rightarrow \infty$ As B is right continuous, we have F ($B\tau n + t$)t ≥ 0 F $\tau n - Ex F$ ($B\tau n + t$)t ≥ 0 F $\tau n - Ex$ +t) $t \ge 0 - F$ ($B\tau + t$) $t \ge 0 - F$ ($B\tau + t$) $t \ge 0 - 0$., $\theta n \in (0, \infty)$. For any two points a, $b \in I$ with a < b, we have $#(A\epsilon \cap (a, b)) \le \epsilon - 1$ ($D + \phi(a)$); hence the Borel-Cantelli lemma yields P[A] = 1. Remark 6.6 In general, convergence in measure does not imply almost everywhere convergence. (ii) Let $m \in N(x, v)$. More precisely, let the pairwise distinct points e1, Then Pnk $\rightarrow Q$, $k \rightarrow \infty$. Then B1 \subset D1 = U \in UD U. f cx+d cd For two linear rational functions f and g, we have Mf $\circ g = Mf \cdot Mg$. Furthermore, $C^{\sim} := \{g^{\sim} : g \in C\} \subset Cb$ (E; C) is an algebra that separates points and is closed under complex conjugation. Nk be the corresponding absolute frequencies. It is easy to check that $\varepsilon \rightarrow |\lambda \varepsilon, N/2|$ is monotone decreasing and that $\varepsilon \rightarrow |\lambda \varepsilon, N/2|$ is monotone decreasing and that $\varepsilon \rightarrow |\lambda \varepsilon, N/2|$ is monotone decreasing. (ii) For almost all $\omega \in \Omega$, the map $I \rightarrow R$, $x \rightarrow f(\omega, x)$ is differentiable with derivative f. Let Bn := A\ ni=1 Ai . 11.2 Snapshot of a voter model on an 800 × 800 torus. Inductively, we get the statement for F (n) since n d \leq (n/ ϵ)n e - n < ∞ for x \geq 0 and $\lambda \geq \epsilon$. Now let H be progressively measurable and bounded. It is easy to check that Y is indeed a random walk with transition matrix p. The product σ -algebra is the smallest σ -algebra is th Theorem 8.12 E[X |F] exists and is unique (up to equality almost surely). In Polish spaces, a partial converse is true. We will come back to this connection in the framework of the martingale convergence theorem that will provide an alternative proof of the Radon-Nikodym theorem (Corollary 7.34). µ is called a • • • • content if µ is additive, premeasure if μ is σ -additive, measure if μ is a premeasure and A is a σ -algebra, and probability measure if μ is a measure and $\mu(\Omega) = 1$. That is, $\mu n - \mu T V \rightarrow 0$ implies $\mu n \rightarrow \mu$ weakly. For any binary prefix code $C = (c(e), e \in E)$, we have $Lp(C) \ge H2(p)$., $r \} 2 / (rdx2)$ is known; see, e.g., [45]. Structure for a counterexample that shows that, in general, the union A U A of two σ -algebras need not be a σ -algebra. 18.4 Speed of Convergence 453 For a practical implementation, there exists an N \in N and mutually disjoint sets F1, . ∞) * P (pi N) = P {p1 · · · pk } -s ∞ n-s n=1 = (p1 · · · pk) -s = k P[pi N]. For n \in N, there exists a $\mu n \in$ M1 (R) such that $\mu * n n = \mu$. Chapter 18 Convergence of Markov Chains We consider a Markov chain X with invariant distribution π and investigate conditions under which the distribution of Xn converges to π for $n \rightarrow \infty$. Due to translation invariance, we have (#BL)-1 Ep [#TL] = r for any L. \blacklozenge Definition 7.6 Let G be a convex set. In this case, we have VT = (XT - K)+ . \clubsuit 2.4 Example: Percolation Consider the d-dimensional integer lattice Zd, where any point is connected to any of its 2d nearest neighbors by an edge. A map $f: E \to C$ is measurable if and only if Re(f) and Im(f) are measurable (see Theorem 1.90 with C ~ = R2). For b > 0, define Hb (p) := - pe logb (pe) e $\in E$ with the convention 0 logb (0) := 0. Apart from the inequalities, the important results for probability theory are Lebesgue's
decomposition theorem and the Radon-Nikodym theorem in Sect. + p(i, j) for i, $j \in E$, and define Yn by Rn (i) = $j \iff Un \in [r(i, j - 1), r(i, j))$. In particular, every characteristic function is uniformly continuous. The analogue of Theorem 13.34 holds for $C \subset Cb$ (E; C). 273 274 281 290 300 224 Contents xiii 14 Probability Measures on Product Spaces . P0 X $\tau N = -N = Rw$, eff ($-N \leftrightarrow N$) Rw, eff ($0 \leftrightarrow -N$) + Rw, eff ($0 \leftrightarrow -N$ + Rw, eff ($0 \leftrightarrow -N$) + Rw, eff ($0 \leftrightarrow -N$ + Rw, eff ($0 \leftrightarrow -N$) + Rw, ef $+ N) Again, since X is transient, we infer)*)*n \rightarrow \infty P0 Xn \rightarrow -\infty = P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup \{Xn : n \in N0 \} < \infty)* = lim P sup$ $\mu * (A \cap B c \cap E) = \mu * (B \cap E) + \mu * (B c \cap E) = \mu * (E)$. $n \rightarrow \infty$ (ii) ϕ is differentiable at 0 with ϕ (0) = i m if and only if (X1 + . (17.32) and Proof (The proof follows the exposition in [100, Section 3]) Since bni , pi ({0}) = (1-pi)ni , conditions (17.31) and (17.32) are clearly necessary for bn1 , p1 ≤ st bn2 , p2 . 21.3. Indeed, there we needed only right continuity of the paths and a certain continuity of the distribution as a function of the starting point, which is exactly the Feller property. Proof "(ii)" This follows from (i) since Theorem 17.11 yields uniqueness of X. Exercise 12.1.1 Let $n \in N$. Then the relative entropy of $\nu \in M1$ (Σ) is $H(\nu \mid \mu) = 1 - m \cdot 1 + m \cdot \log(1 - m)$. How does this work in detail? & Exercise 23.2.7 Let (Xt)t ≥0 be a random walk on Z in continuous time that makes a jump to the left also with rate 12 and a jump to the left also with rate 12 and a jump to the right with rate 12 and a jump to the left also with rate 12 and a jump to the right with rate 12 and a jump to the right with rate 12 and a jump to the left also with rate 12 and a jump to the left also with rate 12 and a jump to the right with rate 12 and a jump to the right with rate 12 and a jump to the right with rate 12 and a jump to the left also with rate 12 and a jump to the right with rate 12 and a jum of upcrossings over a given interval. How stable is weak convergence if we pass to image measures under some map ϕ ? Hence, let F be continuous and linear. $e-\lambda$ (xi) (Poisson distribution) Clearly, ϕ Poi λ (t) = n! n=0 Corollary 15.14 The following convolution formulas hold. 17.7 Stochastic Ordering and Coupling 429 Compute the invariant measure and show the following using Theorem 17.52: (i) If $r \in 0, 12$, then X is positive recurrent. If $f \in C$, then also $|f| = f \infty$ lim pn f 2 /f 2∞ n $\rightarrow\infty$ is in the closure C of C in Cb (E; R). d Differentiating the power series termwise yields dt pt (x, y) = q(x, y). Define the waiting time for the first "success" in the matrix (Xm,n)m,n by Ym := inf n \in C N: Xm, n = 1 - 1. We compute the Laplace transform for these kernels. In particular, for $\alpha > 2$, t $\rightarrow e - |t|$ is not a CFP. By (20.4), we conclude that, for every $x \in E$, P π [X $\in A$] = P choice of countably many sets A1, A2, By Wald's identity (Theorem 5.5), we have E[ST] = E[T] E[X1]; hence '() * Var[ST] = E[T] Var[X1] + ET2 - E[T] 2 E[X1] 2, as claimed. Similarly, we get $Y \ge Y$ almost surely. Lemma 4.6 Let f, g, f1, f2, Manifestly, all three notions of infinite divisibility are equivalent, and we will use them synonymously. Equation (2.9) says that, for any F -measurable Y with E[Y 2] < ∞ ,) *) * E (X - Y) ≥ E (X - E[X |F]) 2 with equality if and only if Y = E[X |F]. By Lemma 15.12(iv) and (ii), ϕ Sn* (t) = ϕ Now 1 - t2 2n n n→ ∞ -→ $e-t 2/2 t \sqrt{n\sigma 2 n}$. Here one needs assumptions on the regularity of the paths $t \rightarrow Xt(\omega)$; for example, right continuity. Then the measure $\mu := pn \ \mu n \in Mf(Rd)$ has $n=1 \ n=1$ characteristic function $\phi \mu = \infty$ (15.3) pn $\phi \mu n$. In this case, (Xi)i $\in I$ and any choice of $x_j \in E$, $j \in J$, $j \in J$, $j * P X_j = x_j$ for all $j \in J = P[X_j + C_j]$ = xj]. + Xn of i.i.d. integrable random variables is n · E[X1]., x d) ∈ Rd and y = 1 d d (y, . Proof Let ε > 0., WK ∈ R. Example 18.6 (Independent coalesce: Let X and Y be independent coalescence) The most important coupling is Markov chains that run independent y until they coalesce: Let X and Y be independent coalescence) The most important coupling is Markov chains that run independent y until they first meet. If we have d = 1 for every state, then the Markov chain is called aperiodic. Definition 9.7 An E-valued stochastic process $X = (Xt) t \in I$ is easy to check that A is a semiring. (ii) Let X be a Markov chain with invariant distribution π . (12.9) $j \in J$ Then the family (Ai)i \in I is called independent given A. Exercise 21.5.1 Use the representation of Brownian motion (Wt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder function (Wt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in
[0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combination of the Schauder functions (Bn,k) to show that the Brownian bridge Y = (Yt)t \in [0,1] as a random linear combinating (Bn,k) to show that the Brownian bridge Y Theorem 15.44 (Central limit theorem of Lindeberg-Feller) Let (Xn, l) be an independent centered and normed array of real random variables. If x, $y \in Zd$ are nearest neighbors (that is, $x - y^2 = 1$), then we denote by $e = x, y^* = y$, x^* the edge that connects x and y. Strictly speaking, this gives the Poisson process only on the time interval (0, 1], but it is clear how to move on: We perform the same procedure independently for each of the intervals (1, 2], (2, 3] and so on and then collect the jumps (see also Exercise 5.5.1). Then E is a π -system that generates 2Ω . Proof We give the proof for σ -algebras only. $|Fn| \leq \max \in \{x0 - n, ..., x0 + n\}$ |F|(x)|, and $A := i = 1, 2, i \in \mathbb{N}$ Hence $f(X) := (f(Xn))n \in \mathbb{N}$ = M + A is the Doob decomposition of f (X). Hence, there is an $\varepsilon > 0$ such that P[X1 < -2ε] > ε . (1.11) 1.3 The Measure Extension Theorem Inductively, we get $\mu * (E \cap Bn) = \text{implies that } 25 \text{ n i} = 1 \mu * (E \cap Bn) = 1 \mu$ 1, We define the conditional probability given B for any $A \in A$ by $\{A \cap B\}$, $P[B] P[A|B] = \{0, if P[B] > 0, (8.1) otherwise. On the other hand, if r is rational, then there exists some <math>n \in Z \setminus \{0\}$ with $e^{2\pi i n} r = 1$. Therefore, $P(x,y)[\tau < \infty] = 1$ for all initial points $(x, y) \in E \times E$ of Z. Then $A \rightarrow \mu(A) := \omega \in A$ pw defines a σ -finite measure on lence (fn)n \in N of measurable maps $\Omega \rightarrow E$ to converge almost everywhere, it is sufficient that one of the following conditions holds., N; hence lim sup $n \rightarrow \infty$ f (yi) – F (yi-1) $\leq 4 \epsilon + f d\mu$. N $\rightarrow \infty$ This completes the proof. We may consider Brownian motion as the canonical process on the ∞)) of continuous paths. The sets A \in A are called events. This property is reflected by the assumption that the jump rate q(x, x + 1) equals x. Takeaways We have adapted the convergence theorems of the last chapter to Lp convergence. Proof See, e.g., [54, Chapter XV.7]. 226 9 Martingales Proof We show that there exist FT -1 -measurable random variables VT -1 and HT such that VT = VT -1 + HT (XT - XT -1). By the strong law of large numbers, we infer Shannon's theorem: 1 1 $n \rightarrow \infty$ - log πn = Yi - \rightarrow H (p) n n almost surely. Strong law of large numbers, we infer Shannon's theorem: 1 1 $n \rightarrow \infty$ - log πn = Yi - \rightarrow H (p) n n almost surely. variables with CFPs ϕ_n , I. For (iii), note that $\tau - s$ peeks into the future by s time units (in fact, { $\tau - s \le t$ } \in Ft + s), while $\tau + s$ looks back s time units. Since E is locally compact, there exists a compact set K \subset E with K $\circ \supset L$ and a ρL , K $\in Cc$ (E) with $1L \le \rho L$, K (x) $\le 1K$. As F satisfies the usual exists and is RCLL. If Y1 $\in L2+\delta$ (P) for some $\delta > 0$, then n ** n $\rightarrow \infty$) E [Xn,1] (2+ δ = n - ($\delta/2$) E [Y1] (2+ δ = n - ($\delta/2$) E [Y1] (2+ δ = - 0., $\omega n \in E$ that the probability of the event [ω_1 , 2 For the integral with respect to the other variable., Ifn) : n $\in N$; f1, n (x1, y1), (x2, y2) > 0. f \in F A A If $\mu(\Omega) < \infty$, then (ii) is equivalent to (iii): (iii) For all $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that $|f| d\mu \leq \varepsilon$ for all $A \in A$ with $\mu(A) < \delta(\varepsilon)$. If in (ii) the measure is a probability measure for all $\omega 1 \in \Omega 1$, then κ is called a stochastic kernel or a Markov kernel. $P\{-n, -n+1, ...\} X^*$ = $P X 0 \in A - n$, $X 1 \in A - n + 1$, ∞ (ii) $n = 0 |An| = \infty$ almost surely. In order to apply the results on finite electrical networks from the last section, we henceforth assume that $A0 \subset E$ is such that $E \setminus A0$ is finite. Proof "(i) = (ii)" Let A, A1, A2, . To this end, we want to apply Kolmogorov's moment criterion. Proof (i) Let m := sup f (SA (x0)). Hence there exist g and h with gn \uparrow g and hn \downarrow h. The basic idea is that it is energetically favorable for particles to be oriented in the same direction. How can we construct a probability space on which all these random variables are defined? For $0 \le b < c \le \infty$ and $t \in R$, let yb,c,t be the linear path from b to b - ibt and let #c,t be the linear path from b - ibt to c - ict, let δb ,t be the linear path from b to b - ibt and let #c,t be the linear path from b - ibt and let #c,t be the linear path from b - ibt to c - ict, let δb ,t be the linear path from b - ibt to c - ict to c. Takeaways Consider a stochastic process that can take only two values at time t + 1 given the full history up to time t. For A \subset E, let $\tau := \tau A := \inf \tau x x \in A$ be the stopping time of the first entrance to A. X is transient if and only if r = 1/2, in which case we have $\mu 1 = \mu 2$. So the stopping time of the first entrance to A. X is transient if and only if r = 1/2, in which case we have $\mu 1 = \mu 2$. MCMC method, it is in fact, at least theoretically, possible to use a very similar method that allows perfect sampling according to the invariant distribution π, even if we do not know anything about the speed of convergence. n→∞ m≥n Brownian Motion and White Noise The construction of Brownian motion via Haar functions has the advantage that continuity of the paths is straightforward. Thus $(n(1 - |\phi_n(t)|2)) \in N$ is bounded for $t \in [-\varepsilon, \varepsilon]$. 7 Lp -Spaces and the Radon-Nikodym Theorem 1.36), we have $\infty v(A) = \lim v n \rightarrow \infty$ Ak $\geq \inf v(An) \geq \varepsilon > 0$. As a direct application of Theorem 16.5, we give a complete description of the class of infinitely divisible probability measures on $[0, \infty)$ in terms of their Laplace transforms. Definition 24.6 Let X be a random measure on E. Proof (i) Assume that A, B \in A. We conclude that Ac \in AI. In a similar way to (viii), we define μ (A) = n i=1 bi f (x) dx. 6. The eigenvalue 1 has the multiplicity 1. Let μ and ν be arbitrary (but nonzero) 3σ -finite measures.3 Then there exist measurable functions g, h: $\Omega \rightarrow (0, \infty)$ with g dµ = 1 and h dv = 1. Usually it is difficult to show in a specific situation that the extension to 2 Ω is impossible. 20.1 Definitions 495 Definitions 495 Definitions 495 Let X = (Xt)t \in I because the extension to 2 Ω is impossible. 20.1 Definitions 495 Definition 20.6 (i) τ is called measure-preserving if *) P $\tau - 1$ (A) = P[A] for all A \in A. Proof Clearly, ft is continuous on R \ {0}. Let X = (Xt)t \in I because the extension to 2 Ω is impossible. 20.1 Definition 20.6 (i) τ is called measure-preserving if *) P $\tau - 1$ (A) = P[A] for all A \in A. Proof Clearly, ft is continuous on R \ {0}. a real-valued, adapted stochastic process with $E[|Xt|] < \infty$ for all $t \in I$ In fact, these sets form a π -system that generates B(Rn) (see Theorem 1.23). $\in L1$ (μ) be nonnegative and such that lim fn d μ $n \rightarrow \infty$ exists. Clearly, M = M is the trace σ -algebra of M Hence $M(E) \in M$. Let $p, p \in [0, 1]$ with p < p. Here we used $F\tau \supset F\sigma$, the tower property and the monotonicity of the conditional expectation (see Theorem
8.14)., X(n-1) - X(n) |X(n)| and that it equals the (unconditional) distribution of the ordered values of X1,. The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurat at the date of publication. Define $Z := \lim \sup n \to \infty 1$ Sn . Remark 21.7 The statement of Theorem 21.6 remains true if X assumes values in some Polish space (E,) since in the proof we did not make use of the assumption that the range was in R. If we let f = 1E in (7.8), then we get $\mu(E) = 0$. Let $F \subset A$ be a σ -algebra and let κX , F be a regular conditional distribution of X given F. Proof (i) Let $AJ = \sigma \times Ej : Ej \in Ej \cup \{\Omega j \}$ for every $j \in J$. Also assume that we follow a bounded trading strategy that cannot use future information. Theorem 5.5 (Wald's identity) Let T, X1, X2, . If only (i) and (ii) hold and)x, $x^* \ge 0$ for all x, then) \cdot , \cdot^* is called a positive semidefinite symmetric bilinear form, or a semi-inner product. Furthermore, $(An)n \in N$ is an independent family with the property $\infty \infty 1 * P[An] = 6 = \infty$. For $n \in N$, define $An := \{[\omega 1, . The monotonicity of <math>\mu * now \mu * (E) = \mu * (E \cap Bn) + \mu * (E \cap Ac) = n \mu *$ pw (i, j) = wi+, if j = i + 1, 0, else. Then there exists an increasing sequence K1 C K2 C K3 C . 16 1 Basic Measure Theory Definition 1.34 Let A, A1, A2, . Define Yn (ω) := -log(pXn (ω)). Thus it comes in very handy that it is sufficient to check measurability on a generator of A by the following theorem. For some applications, however, a decomposition in trigonometric functions is preferable. lim sup an $n \rightarrow \infty$ k=1 See [22]. be real random variables with $Xn \Rightarrow X$. \blacklozenge Example 20.9 (Rotation) Let $\Omega = [0, 1)$, let $A = B(\Omega)$ and let $P = \lambda$ be the Lebesgue measure. \blacklozenge If f, g : $\Omega \rightarrow R$ with f (ω) $\leq g(\omega)$ for any $\omega \in \Omega$, then we write f $\leq g$. Proof " \Rightarrow " Let ν be totally continuous with respect to μ . (i) (ii) (iii) (iv) $E[Xi] = \mu i$ for all i = 1, . < tn, the family (Nti – Nti–1, i = 1, . Therefore, since $pn+k(x, x) \ge pn(x, y) pk(y, x)$, we have $G(x, y) = \infty$. k=1 See Fig. Show (using Theorem 4.15) that $xf(x) \lambda(dx)$. Theorem 8.5 Let A, $B \in A$ with P[A], P[B] > 0., Xn) and Sn := X1 + . We give an introduction to the basic concepts and then study certain examples in more detail. 1 1 11 18 36 45 2 Independence . Taylor's theorem (Lemma 15.31) yields eit x - 1 - itx = -t 2x2 + R(tx) 2 15.5 The Central Limit Theorem 361 with $|R(tx)| \le 16$ $|tx|^3$. Further, let X1 , X2 , . 7 Lp -Spaces and the Radon-Nikodym Theorem 176 Theorem 7.29 (Uniqueness of the density) Let ν be $] \rightarrow 0$. Let $\alpha > 0$ and let L be a Poix random variable. Klenke, Probability Theory, Universitext, 147 148 6 Convergence Theorems fz (ω) = d(f (ω), z) are also measurable. Note that $\beta = 0$ (x1)/N and P[Xk+1 = xk+1 |X1 = x1] = g0 (x1)/N and P[Xk+1 = xk+1 |X1 = x1]. Hence $\gamma = 0$ (x1)/N and P[Xk+1 = xk+1 |X1 = x1] = g0 (x1)/N and P[Xk+1 = xk+1 |X1 = x1]. $E[f(X)] E[g(Y)] + E[f(Y)g(Y)] - E[f(Y)] E[g(X)] = 2 \operatorname{Cov}[f(X), g(X)]$. lim i=1 In particular, Ap \cap Aq = \emptyset if p = q and thus (Berp) N \perp (Berq) N \perp (Berq) N \perp (Berq) N (μ) is called uniformly integrable if sup inf $0 \le g \in L1(\mu)$ f $\in F | f | -g + d\mu = 0$. N $\in N \in N$ a, $b \in Q + 0 \le at Xs$ (ω) t (ω) = 0. To every canonical triple, by (16.8) there corresponds an infinitely divisible random variable. Defining Sⁿ = X¹ 1 + . \cap Aik), {i1 ,...,ik } $\subset \{1,...,n\}$ k=1 $\mu(A1 \cap ..., We now extend the result to a larger class of ranges for Y . (21.28) n=1$ Theorem 21.30 d is a complete metric on $\Omega := C[0, \infty)$ that induces the topology of uniform convergence on compact sets. AN $\tilde{}$ fn) $- \rightarrow 0$ for all N \in N. Lemma 6.1 Let f, g : $\Omega \rightarrow E$ be measurable with respect to A - B(E). In particular, we have Ex [Yt] = 2x t + x 2, Ex [Yt3] = 6x t 2 + 6x 2 t + x 3, Ex [Yt4] 24x t 3 + 36x 2 t 2 + 12x 3 t + x 4, Ex [Yt5] = 120x t 4 + 240x 2 t 3 + 120x 3 t 2 + 20x 4 t + x 5, Ex [Yt6] = 720x t 5 + 1800x 2 t 4 + 1200x 3 t 3 + 300x 4 t 2 + 30x 5 t + x 6. Proof For p = ∞ , the equivalence of (i) and (ii) is a simple consequence of the triangle inequality. If μ is an invariant measure, then the equations for μ p = μ read μ ({n}) = $pn-1 \mu(\{n-1\}) \mu(\{0\}) = \infty$ for $n \in N$, $\mu(\{n\})(1-pn)$. We infer ; ; ; (f-g)1[-K,K]d; $< \varepsilon \infty$ and ; ; ; (f-g)1d; $\sim \le f^{\infty} + \varepsilon = f \infty + \varepsilon$. By the preceding corollary, $(fF : F \in I)$ is uniformly integrable with respect to μ . Hence we get $m\beta,h \approx h h = \beta - 1 - 1$ T - Tc for $T \to \infty$, (23.24) where the Curie temperature Tc = 1 is the critical temperature for spontaneous magnetization. If the latter were the case, a ramification into more than two values in one time step would be possible. n! Corollary 15.33 (Method of moments) Let X be a real random variable with $\alpha := \lim \sup n \to \infty 1$) n *1/n E $|X| < \infty$. Assume that for any finite J \subset I, any choice of Aj \in Aj and for all $j \in J$, P ') * Aj A = P Aj A j \in J almost surely. Now choose a finite subcovering {U1, . Furthermore, $\delta(E) \subset \sigma$ (E). However, τr is not mixing: Since r is irrational, there exists an Hz \in C with Hz (z) = Hz (x) = 0. By Fatou's lemma (Theorem 4.21) with 0 as a minorant, we thus get $|\text{fnk}| \, \text{d} \mu < \infty$. When a particle dies, it has two offspring. Takeaways Consider the conditional probability of some event B given a σ -algebra. max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum and maximum differ at most by a factor N: max i=1,...,N ϵ -0 Proof The sum A = B(E)@[0,∞). \$ 594 23 Large Deviations 23.2 Large Deviations Principle The basic idea of Cramér's theorem is to quantify the probabilities of rare events by an exponential rate and a rate function. ♦ Reflection Maybe it comes as a surprise that in the definition of stochastic convergence, the set A of finite measure pops up. The resulting measure is called Lebesgue measure (or sometimes Lebesgue-Borel measure) λ on n R, B(Rn). 12.1 Exchangeable Families of Random Variables .. The relation (23.24) is called the Curie-Weiss law. The following frequencies for letters in German texts are taken from [11, p. If g = 1A1 × A2 for some A1 ∈ A2, then clearly Ig (ω 1) = 1A1 (ω 1) $\kappa(\omega 1, A2)$ is measurable. This is a material dependent constant (chromium bromide (CrBr) 37 Kelvin, nickel 645 K, iron 1017 K, cobalt 1404 K). If in (v) the sum $\omega \in \Omega$ p ω equals one, then μ is a probability measure. Proof The p.g.f. of the Poisson distribution is $\psi(z) = e\lambda(z-1)$ (see (3.4)). Theorem 23.13 (Sanov [150]) Let X1, X2, Then Yn $\rightarrow X$. Exercise 1.5.4 Show that $F: R2 \rightarrow [0, 1]$ is the distribution function of a (uniquely determined) probability measure μ on (R2, B(R2)) if and only if (i) F is monotone increasing and right continuous (ii) F ((-x1, y2)) $\rightarrow 0$ and F (x) $\rightarrow 1$ for all y1, y2 $\in R$ and for x = (x1, x2) $\rightarrow \infty$, (iii) F ((y1, y2)) - F((y1, y2)) - F((y1, y2)) + F((y1, y2)) + F((y1, y2)) + F((y1, y2)) - F((y1, x^2 (1) ≥ 0 for all $x_1 \leq y_1$ and $x_2 \leq y_2$. $n \rightarrow \infty$ n Remark5.13 The strong law of large numbers implies the weak law., Xtnk) \Rightarrow (Xt1, Examples for
measure preserving dynamical systems are stationary stochastic processes, i.i.d. random variables, rotations and so on. For a proof of (ii) \Rightarrow (i) see, e.g., [155, Theorem III.4.3]. property of Brownian motion) If B is a Brownian motion and if K = 0, then (K - 1 BK 2 t) t ≥ 0 is also a Brownian motion. Hence C0 is dense in Cb (E; R). Now if X is not a martingale, then in some cases, we can replace X by a different process X that is a martingale and such that the distributions PX and PX are equivalent; that is, have the same null sets. On the other hand, $n \rightarrow \infty$ in this case, there is also some kind of recurrence property, namely Sn /n $\rightarrow 0$ almost surely by the ergodic theorem. Hence $\omega = (\omega_1, \omega_2, ..., \omega_1)$ the is measurable with respect to A2. Now let X, Y, Z be independent real random variables with characteristic functions $\phi X = \phi Y = \phi$ and $f \omega^2 1 = f - 1$ (A) \in A2 and $f \omega^2 1 = f - 1$ (A) = f - 1 $D \phi Z = \psi$. We interpret [$\omega 1$, 1 2 Expanding the cosine function in a Taylor series, we get cos(ti) = 1 - 2 ti + O(ti4); 1 hence 1 - $\phi(t) = 2D t22 + O(t42)$. PX = P-X if and only if ϕ is real-valued. Since (14.6) and (14.7) hold for f - and f + , by Remark 14.17 and 14.18 this is true for f also. t $\in \mathbb{R} \vee U$ under the assumption that F = F^{*} holds, n/2 Dn converges in distribution to M. Similarly, (14.7) holds for f = 1A. ($\beta > 1$) where so-called spontaneous magnetization occurs (that is, magnetization occurs (that is, magnetization without an exterior field). Hence X-1 (α (E)) $\subset \sigma$ (X-1 (E)). It is the price multiplied by the number of stocks in the portfolio. This follows from Corollary 8.22 and the fact that $X - n = E[X0 \quad F - n]$ for any $n \in N0$. Note that $Z\infty$ is the only choice since $\sigma 2 > 0$. Then $1\{x \leftarrow \rightarrow p y\}$ is a random variable. However, it is not too hard to give an estimate that shows that there a c = cD such that $p^2(0, 0) \le c n - D/2$, which implies $G(0, 0) \le \infty exists - D/2$ c n = 1 $n < \infty$ (see, e.g., [53, page 361] or [59, page 361] o Example 6.31]). 2.3). On the other hand, if μ is a Lebesgue-Stieltjes measure, this is $\mu = \mu F$ for some F, then #{ $n \in N : xn \in (-K, K]$ } = F (K) - F (-K) < ∞ for all K > 0; hence (xn) n \in N does not have a limit point. Let X1, X2, Now let A \subset E. Let E be the exchangeable σ -algebra and let T be the tail σ -algebra. \blacklozenge 13.3 Prohorov's Theorem 291 Recall that a family F of measures is called weakly relatively sequencies in F has a weak limit point (in the closure of F). Similarly, for $x \in E N$, denote x = (x(1), x(2), . In the case where Xn+1 - Xn takes three values, the system has three equations and is thus overdetermined. Define A2L,0 in a similarly way to A2L; however, we now consider all edges $e \in EL p p$ as closed, irrespective of whether Xe = 1 or Xe = 0. Then the family (P $\xi n (X)$) $n \in N$ of distributions of empirical measures satisfies an LDP with rate n and rate function I $\mu := H (\cdot | \mu)$. 19.3 Electrical network on Z2., Xn = kn] $\mu({x})n\nu({x}) = #An (\nu) \exp n \nu(dx) \log \mu({x}) = #An (\nu) \exp n \mu({x}) = An (\nu)$ + H ($\nu |\mu$)]. 1 Furthermore, by Theorem 12.10 (with k = n and $\phi(X1, Let \tau n = inf\{t \ge 0 : Xt = n\} = Sn$ for $n \in N$. 46 1 Basic Measure Theory Theorem 1.104 For any distribution function F, there exists a real random variable X with FX = F. If c0, Theorem 1.104 For any distribution function F, there exists a real random variable X with FX = F. If c0, Theorem 1.104 For any distribution function F, there exists a real random variable X with FX = F. If c0, Theorem 1.104 For any distribution function F, there exists a real random variable X with FX = F. If c0, Theorem 1.104 For any distribution function F and $\phi(X1, Let \tau n = inf\{t \ge 0 : Xt = n\}$ unlock the information. Reverse Address Lookup A reverse address lookup is another type of search you can do if you only have part of the information about the number you need to find. $n \rightarrow \infty$ Thus $q \in F$. Let $TN \rightarrow \infty$ with $E[XTN] \ge E[X0]$ for all $N \in N$. $n \rightarrow \infty$ (13.7) If F, F1, F2, . For $s \ge 0$, define the Mellin transform of PX by mX (s) = E[Xs] current flow., N - 1, be the Nth roots of unity and let the corresponding (right) eigenvectors be $x k := \theta k0$, $\theta k1$, 42.3 Kolmogorov's 0-1 Law With the Borel-Cantelli lemma, we have seen a first 0-1 law for independent events. 20.6 Entropy 511 Then the entropy of Pn is (using stationarity of π in the third line) H (Pn) = $-p(0, x) \log(p(0, x)) x 0$ $(1-g)(\mu+\nu)$; that is, the measure with density $(1-g)(\mu+\nu)$; that is, the measure before a fraction of 2% of the measure $(1-g)(\mu+\nu)$; that is, the measure with density $(1-g)(\mu+\nu)$; that is, the measure $(1-g)(\mu+\nu)$; the measure $(1-g)(\mu$ production is defective. (ii)., $Mn,m = km = n! km pk1 \cdots pm$. By Taylor's formula, for every $t \in (-\epsilon, \epsilon)$, $n-1 |t|2n-1t 2k (2k) u(0) \le u(t) - (2k)! (2n-1)! k=0$ sup $u(2n-1) (\theta t)$. As 15.4 Characteristic Functions and Moments 351 $E[|X|k] < \infty$ by assumption, the dominated convergence theorem implies $h \rightarrow 0 E[Yk(t, h, X)] \rightarrow E[(iX)k \text{ eit } X] = 0$ $\phi(k)$ (t). Together with (23.19), this implies (23.17). = exp - $\lambda + 1/t$ However, the function $\psi(x) = 0$ (wm + wn) - x $\geq c$. 8.3 Regular Conditional Distribution 209 A separable topological space whose topology is induced by a complete metric is called a Polish space., xk) = (x1, x1 + x2, A similar argument for the right-hand side yields continuity of ϕ at x. (i) The family (fZ : Z \in Z) is uniformly integrable in L1 (μ) and $\nu(\Omega)$ for any Z \in Z. A Exercise 20.3.2 Let p = 2, 3, 5, 6, 7, 10, Exercise 8.2.1 Show the assertions of Remark 8.16. F (x) = inf F is monotone increasing, F is right continuous and monotone increasing, F is right continuous and monotone increasing. Theorem 5.23 (Glivenko-Cantelli) Let X1, X2, . (i) $\mu = w$ -lim μn . $n \rightarrow \infty$ Then $gn \rightarrow 0$ in measure, and (gn $n \in \mathbb{N}$ is uniformly integrable since $gn \leq n \rightarrow \infty$ p 2p (|fn|p + |f|p). For $Z \in Z$, define a function $fZ : \Omega \rightarrow \mathbb{R}$ by $fZ(\omega) = C \in Z: \mu(C) > 0 \nu(C)$ 1C (ω). Lemma 11.18 W is a martingale. In fact, if for $0 \in J \subset I$ finite, we define PJ as the projective. This is precisely what memory is and is thus in contrast with the Markov property of X. That is, $p(e_1, e_2) = pe_11$ and $e_2 \in E_2$ $p(f_1, f_2) = pf_2 2$ for all $e_1 \in E_1$ $f 1 \in E 2$. 20.5 Mixing ... Hence $\infty X = X + n n n = 1 n = N$ Yn n = 1 converges a.s. " \Rightarrow " Assume that $\infty n = 1$ Xn converges a.s. (i) (otherwise, by the Borel-Cantelli lemma, |Xn| > K infinitely often, contradicting the assumption). (ii) For transient X, there can be more than one invariant measure. Choose three pairwise distinct points x 1, x 2, x 3 \in BL \ BL-1 with Pp [Fx 1, x 2, x 3] > 0. Now assume that Q is absolutely continuous with respect to P. Let x, y \in E, and let (X, Y) be a successful coupling. with Tn ~ expn . For every n \in N, let an be the leading digit of the p-adic expansion of q n . Exercise 1.2.1 Let A = {(a, b] \cap Q : a, b \in R, a \leq b}. Clearly, E[T ∞] = n=0 1/wn < ∞ ; r < ∞] = 1. & Exercise 15.2.3 Let X be a real random variable with characteristic function ϕ . $n \rightarrow \infty x \in E$ N Now let f1, . As μ is regular (Theorem 13.6), there is a compact set $K \in with \mu(E \setminus L) < \varepsilon$. μ -almost 280 13 Convergence of Measures (ii) For any $\varepsilon > 0$, there is a compact set $K \in with \mu(E \setminus L) < \varepsilon$. (14.4) J CI countable Hint: Show that the right-hand side is a σ -algebra. $2 = \delta/R2 = 27$ and R R 5 125 Step 8. Then the following statements
are equivalent. For irreducible chains, all states have the same period. Proof First assume that (Nt, t ≥ 0) is a Poisson process with intensity $\alpha \geq 0$. Hence, it is enough to show (14.8) for sets of this type. (21.42) Proof The exact formulas for the first six moments are obtained by tenaciously computing the right-hand side of (21.40). Later, with a lot of additional effort, we will achieve a sharp bound in Theorem 22.11. We come back to Polish spaces in the context of convergence of measures in Chap., $\sigma(2n, 6)$. Hence also $\beta(A1 \cup A2) = \sup_{n=1}^{\infty} \beta(A1 \cup A2)$ $\alpha(C): C \in C$ with $C \subset A1 \cup A2 \leq \beta(A1) + \beta(A2)$. Thus also $X = \inf n \in N$ is A - B(R)-measurable and is hence a random variable. In particular, if f = g3 almost then $f d\mu = g d\mu$. 4.3 Let $\nu(dx) = \pi 2 1 - x 2 1[-1,1]$ (x) dx. In fact, in this case, the reflection principle can be derived also in an elementary way via a bijection that changes the signs of those Yi with i > τ . $\sqrt{In the following, let i = -1 be the imaginary unit. <math> = 1$ be the projection of Ω to the nth coordinate., Xk) = E (F dvn,k(X). Then there exists a random variable Y : $\Omega \rightarrow [0, 1]$ such that, for all finite J $\subset N$, *) P Xj = 1 for all j \in J Y = Y #J. (We have done this already for d = 1, see Exercise 13.3.4 for d \geq 2.) In a second step, the statement is lifted to sequence spaces RN. We infer that X is recurrent if and only if $3-2 t 2 \tau y1$. (ii) The series defining hab converges in L2 ([0, 1], λ). (ii) Infer the optional sampling theorem for right continuous supermartingales by using the analogous statement for discrete time (Theorem 10.11); that is, $X\sigma \ge E[X\tau | F\sigma]$. Define have E U the F ∞ -measurable events 1 0 a, b C = lim inf Xn < a \cap lim sup Xn > b C U a, b = ∞ n $\rightarrow\infty$ n $\rightarrow\infty$ and C = C a, b. We have to show that PJ \circ (XL L l for some l = 1, . E fn (tX) X2n \le |t| $\theta \in (0,1]$ θ |t| $\theta \in (0,1]$ Now Fatou's lemma implies)) *) * * E X2n = E fn (0)X2n \leq lim inf E fn (tX)X2n t $\rightarrow 0$ \leq lim inf gn (t) = 2n u(2n) (0) $< \infty$. On the n space (Ω, A, P) := ×i=1 Ω i , ni=1 Ai , ni=1 Pi , the coordinate maps Xi : $\Omega \rightarrow \Omega$ i are independent with distribution PXi = Pi . (iii) (Positive definiteness))x, x* > 0 for all x \in V \ {0}. $\pi(x-i) \pi(y-i)$ Thus the Gibbs sampler is a reversible Markov chain with invariant measure π . Corollary 7.22 (L2 (μ),) · , · *) is a real Hilbert space. Next we show that Y has a continuous for any $\gamma \in (0, 1]$. Then $F \circ X = 1A$. Then $X = (Xn)n \in N0$ is a square integrable martingale with respect to $F = \sigma(X)$ (why?) and *) *) 2 2. The vector $p = (p\omega)\omega\in\Omega$ is called a probability vector. The validity of (2.2) follows as in Example 2.1(i). In many cases of interest, these quotients are easy to compute even though $\pi(x)$ and $\pi(y)$ are not. Clearly, $0 \in G$; hence $G = \emptyset$. lim sup sup Fn (x) - F (x) $\leq N N n \rightarrow \infty$ the claim follows. A stable distribution is characterised by its index α and a skewness parameter (and, of course, a scale parameter); see Remark 16.23. 17.1 Definitions and Construction). Reflection Give an example of a current that is not an electrical current. Rather, F β is asymmetric and has a global minimum m β , h with the same sign as h. νa has a density with respect to μ , and $d\nu$ be σ -finite measures on (Ω , A). By the Ionescu-Tulcea theorem (Theorem 14.35), the projective limit P := {-n -n+1,...} exists. On this tree, random walk is transient. Furthermore, by the $n \rightarrow \infty$ ny 0, we have Similarly, lim inf lim 1 1 log Pn (($-\infty$, x)) = 0 = -I (0). In this section, we derive two simple criteria that prepare us for important applications such as the law of large numbers (Chap. A Theorem 6.28) = 0 = -I (0). In this section, we derive two simple criteria that prepare us for important applications such as the law of large numbers (Chap. Theorem 6.28) = 0 = -I (0). (Differentiation lemma) Let $I \subset R$ be a nontrivial open interval and let $f: \Omega \times I \to R$ be a map with the following properties. By Vd denote the set of monotone increasing, bounded right continuous functions on Rd. Again, by Theorem 6.19, it is enough to show that $E[f(Yn)] \leq C$ for every $n \in N$. In a second step, this random variable will be transformed by applying the inverse map F - 1: Let $\Omega := (0, 1)$, $A := B(R) \Omega$ and let P be the Lebesgue measure on (Ω, A) (see Example 1.74). By Exercise 17.7.3 Let $n \in N$, $p \in (0, 1)$ and $\lambda > 0$., Xsn = in)*)* = $E E[1{Xt = i} Sn] 1A = E E[1{Xt = i} Xsn] 1A$)*)* P[Xt = i Xsn = in] 1A = P Xt = i Xsn = in P[A]. Show that $vt = lims \rightarrow t vs$ for all t > 0. Proof The proof is based on a Taylor expansion of the logarithm, $|\log(z) - (z - 1)| \le |z - 1|^2$ for $z \in C$ with |z - 1| 0 with $|\phi(t)| > 12$ for all $t \in [-\varepsilon, \varepsilon]$. Then, by the monotone convergence theorem, $\mu 1 (d\omega 1) \cdots \mu(A)^{-1} = \le \mu n (d\omega n)$ 1A (($\omega 1$, . Denote by hn the entropy of Pn. That is, there exists a square integrable martingale X that converges almost surely. Thus $g \in E + with g \leq f$. 18.3 Markov Chain Monte Carlo Method 447 • Each atom $i \in A$ has a magnetic spin $x(i) \in \{-1, 1\}$ that either points upwards (x(i) = +1) or downwards (x(i) = -1). That is, we have p(hg, hf) = p(g, f) for all $h, g, f \in G$. Now let U be an open set that contains f, but with $fn \in U$ for infinitely many $n \in N$. + Zn for $n \in N$ and $Xn := X^{n} Sn + X^{n} - Sn$. In Step 2, we have shown already that N does not assume a finite value larger than 1. Takeaways Let E be countable and let p be a stochastic matrix on E. Hence (18.2) holds with ImJ ImJ and Lx, y := m - d. Hence, let $p \in [1, \infty)$. In addition, define hx := f. If $n \in N$ is not a multiple of d, then pn (x, x) = 0. The n-step transition probabilities p(n)(x, y) := Px [Xn = y] can be computed as the n-fold matrix product p(n)(x, y) = pn(x, y), where $pn(x, y) = z \in E$ and where p0 = I is the unit matrix. Clearly, $\nu(\emptyset) = 0$. (iv) The second equality follows from (iii) with Y = E[X | G] and X = 1. This is indeed true since for t > s, the random variables Xs and Xt - Xs are independent; hence Cov[Xs, Xt - Xs] + Cov[Xs, Xs] = Var[Xs]] = s. Here the problem that arises when (E[X1], . Then there exists a unique σ -finite measure $\mu \otimes \kappa$ on ($\Omega + \alpha 2$) + $\kappa(\omega 1 + \alpha 2) = \kappa(\omega 1 + \alpha 2)$ (Lebesgue integral) Let λ be the Lebesgue measure on Rn and let $f : Rn \rightarrow R$ be measurable with respect to B * (Rn) - B(R) (here B * (Rn) - B(R)) (here B * (Rn) - B((Rn) is the Lebesgue σ -algebra; see Example 1.71) and λ -integrable. Then (Xn)n \in N is i.i.d. BerZ n $\rightarrow \infty$ distributed given Z. The fundamental question is: For which values of p is there a connected infinite system of tubes along which water can flow? In most cases, only the distribution of a random variable is of interest but not the underlying probability space. Let Sk := X1 + . 4 6.2 Uniform Integrability 153 Exercise 6.1.4 Let (Xi)i < N be independent, square integrable random variables with E[Xi] = 0 for all i < N. In the second section, we investigate briefly which subclass of the infinitely divisible measures on R shares this property. 524 21 Brownian Motion Recall that a stochastic process (Xt) t \in I is called a Gaussian process if, for every $n \in N$ and for all t1, . If $x \in \partial f - 1$ (D) \cap Be(δ) (x) and $z \in f - 1$ (D) \cap Be(δ) (x). 13.2 Weak and Vague Convergence 287 Corollary 13.24 Let X, X1, X2, . 1.1 Classes of Sets 3 Remark 1.5 Sometimes the disjoint union of sets is denoted by the symbol . a 2 t 2 a 2 t 2 Here we used the fact that by the addition theorem for trigonometric functions $1 - \cos(x) = \sin(x/2)2 - \cos(x) = 2 \sin(x/2)2 - \cos(x) = 2 \sin(x/2)2$. Now assume that X is a fair game (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally
bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a martingale) and H is locally bounded (that is, a hold? Mayer, Mario Oeler, Marcus Schölpen, my colleagues Ehrhard Behrends, Wolfgang Bühler, Nina Gantert, Rudolf Grübel, Wolfgang König, Peter Mörters, and Ralph Neininger, and in particular my colleague from Munich Hans-Otto Georgii. Then gx is monotone increasing and there exist the left-sided and right-sided derivatives D $g_x(y) = \sup\{g_x(y): y < x\} \neq x$ and $D + \phi(x) := \lim g_x(y) = \inf\{g_x(y): y > x\}$. Reflection Where in the previous proof did we exploit the \cap -stability? In fact, weak convergence can be characterized by this property. $-1 - 1 : (\Omega, A) \rightarrow \text{In particular}, \{t : F(t) \le x\} = F(x)$. Since F(x) = F(x). Since f(x) = F(x). \in G, we have ν s (A) = ν (A) - A f du \geq 0 for all A \in A, and thus also ν s is a finite measure. The statement still holds if Rd is replaced by a locally compact Abelian group. We say that μ is (i) monotone if μ (A) $\leq \mu$ (B) for any two sets A, B \in A with A \subset B, 12 1 Basic Measure Theory (ii) additive if μ n Ai i=1 = n μ (Ai) for any choice of finitely many mutually i=1 n Ai \in A, disjoint sets A1, For example, consider d = 2 and μ 1 = 1 1 $\delta(0,0) + \delta(1,1)$ 2 and μ 2 = 1 1 $\delta(1,0) + \delta(0,1)$. The situation is similar to that of Wright's model; however, now in each time step, only (exactly) one individual gets replaced by a new one, whose type is chosen at random from the whole population. Furthermore, the integral is a cornerstone in a systematic theory of probability that allows for the definition and investigation of expected values and higher moments of random variables. Dividing both sides by p-1 f + gp yields (7.2). Hint: Do not try a direct computation! & Exercise 15.5.3 Let X1 , X2 , . . . processes at all. If X is a discrete Markov chain, then (Px) $x \in E$ is determined by the transition matrix $p = (p(x, y))x, y \in E := (Px [X1 = y])x, y \in E := (Px [X1 = y$ canonical process on (Ω, A) . In this case, the family (Xi)i \in I is independent if and only if, for any finite J \subset I fJ (x) = fj (xj) for all $x \in RJ$. If now A $\cap E = \emptyset$ and $\mu(A) = 0$, then 1A d $\mu = 0$. \blacklozenge n=1 n=1 Example 2.9 We roll a die only once and define An for any n \in N as the event where in this one roll the face showed a six. Lemma 21.44 For the branching process with critical geometric offspring distribution, the nth iterate of the probability generating function is $\psi(n)$ (s) = n - (n - 1)s. Indeed, this follows by elementary combinatorics since for any choice x1, Now let x- < 0 < x+. By Theorem 13.11(ii), Cc (R3d) is a separating class for Mf (Rd). Note that Xn depends only on X1, Define X := 2 - 2 Tk+1 }. Hint: Kolmogorov's three-series theorem (Theorem 15.51). We use this idea to give a different proof for the fact that simple random walk on Zd is recurrent if and only if $d \le 2$. Then Corollary 7.8 holds with I replaced by G. (ii) (Symmetry))x, $y^* =$)y, x^* for all x, $y \in V$. This completes the proof of Prohorov's theorem. Proof The strategy of the proof consists in constructing a measurable version of the distribution function of Y by first defining it for rational values (up to a null set) and then extending it to the real numbers. $- < \infty$ or $R + < \infty$, then (agreeing on Theorem 19.33 (i) If $Rw w^*$) $n \rightarrow \infty P0 Xn \rightarrow -\infty = + Rw - + Rw + Rw$ and *) $n \rightarrow \infty P0 Xn \rightarrow +\infty = \infty$ $\infty = 1$) - Rw - +. Proof (i) and (ii) are trivial. Then Xt is measurable with respect to Ft., in), the vector Y = (Xi1, .430 17 Markov Chains There are many concepts to order probability measures on R or Rd such that the "larger" one has a greater preference for large values than the "smaller" one. Proof (We follow the proof in [59, Theorem 3.34]) We must show that for any $n \in N$ and any sequence 0 = t0 < t1 < . (ii) \Rightarrow (i) This is trivial. By the triangle inequality (Theorem 4.20), we conclude)*)**) $E E[X | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]_2 = \lim E E[|X| \land N | F]$ necessary prerequisites for a systematic probability theory. The latter property can be achieved if the marginals are stochastically ordered. \blacklozenge Reflection Consider the function f (x) = 1/x, x $\in [-1, 1] \setminus \{0\}$, f (0) = 0. n $\in \mathbb{N}$ Hence 3 n $\in \mathbb{N}$ f dµ = $\gamma \leq \nu(\Omega)$. Show that A = A $\subset \Omega$: A is countable or Ac is countable . By the central limit theorem, for σ n) n * $n \rightarrow \infty t > s \ge 0$, we have L St - Ssn - $\rightarrow N0, t - s$. By the reflection principle, *) P[$\zeta \le t$] = P Bs = 0 for all $s \in [t, 1] \infty$) * P Bs = 0 for all $s \in [t, 1] \infty$)
* P $supn \rightarrow \infty Xn = 1$ almost surely. For p > 12, there exists a unique infinite connected component of open edges (Theorem 2.47). A particular case is that of a graph (E, K) where all edges have the same conductance, say 1; that is, C(x, y) = 1 ($x, y* \in K$). Show that X is exponentially distributed if and only if P[X > t + s | X > s] = P[X > t] for all s, $t \ge 0$. We may assume that $L = J \setminus \{j\}$ finite. $k \to \infty$ It remains to show that there exists a measure μ on (E, E) that satisfies (13.12). Assume $A = \{x0, x1\} \downarrow \emptyset$ but $\mu(An) = \infty$ for all $n \in N$., Xn,n such that X = Xn, 1 + . Clearly, $\#\Pi0,m \leq 2d \cdot (2d - 1)m - 1$ since there are 2d choices for the first step and at most 2d - 1choices for any further step. We say that X jumps with rate q(x, y) from x to y if the following limit exists: $q(x, y) := \lim t \downarrow 0 \ 1 \ Px [Xt = y]$. If $A \in E$, then 279 3 f dµ1 = µi (A) = sup µi (K) : K \subset A is compact since the Radon measure µi is inner regular (i = 1, 2). , Xn) given ΞN ? This is the cornerstone for the proof of de Finetti's theorem. Formally, we define $B_j = A_j$ and $B_j = A_j$ for $j \in J$ and $B_j = E$ for $j \in I$, be random variables with values in E. Definition 9.24 Let (Ω , F, P) be a probability space, $I \subset R$, and let F be a filtration. 19.1). , Xn) is almost surely in one of those flat pieces. For any $C \in Zm$, either $C \cap B = \emptyset$ or $C \subset B$. 19.5 Scheme of the first three steps (two stages) of the graph from Example 19.31. Lemma 1.51 An outer measure $\mu *$ is σ -additive on M($\mu *$). In Exercise 2.1.2 it was shown that the conclusion of the Borel-Cantelli lemma still holds under this weaker assumption. It remains to show $\nu s \perp \mu$. Xk = xk] = gk (xk+1) N -k for k = 1, . The answer is given by the source coding theorem for which we prepare with a definition and a lemma. Proof "(ii) \Rightarrow (i)" Assume that (i) does not hold. Hence (H · Y) $n \ge (b - a)Una, b$ for all $n \in N$. (ii) " \Rightarrow "Assume that (Ω, A, P, τ) is ergodic. $\epsilon \downarrow 0$ For any $n \in N$ and $\epsilon > 0$, we have $P[N2 - n \ge 2] \ge 2 - n/\epsilon!$ $P[N\epsilon \ge 2] - 2 - n$ disjoint, and assume that $B = Bn \in A$., $bk \} = B1 + .$ In addition, we have more freedom if, as in the last proof, we want to express X as a sum of independent random variables Xk. By (23.14), we have $H(x|\lambda) = log(\#\Sigma) - H(x)$, where $H(x|\lambda) = log(\#\Sigma) - H(x)$. first show that Pp[N = m] = 1 for some m = 0, 1, . The n-dimensional volume of such a rectangle is $\mu((a, b]) = n$ (bi – ai). Show the following. Then E[X | F] is the orthogonal projection of X on L2 (Ω, F, P). Denote by $Dn : \Omega \to \{-1, 1\}, \omega \to \omega n$ the result of the nth game (for $n \in N$). Thus $\alpha(C) \le \alpha(C1 \cup C2) \le \alpha(C1$ Indeed, for $n \in N$, let $Xn ((\omega 1, \omega 2, ., n \rightarrow \infty$ Proof This is obvious since Ceff $(x1 \leftrightarrow \infty) = C(x1)$ inf $pF(x1, A0) : |E \setminus A0| < \infty, A0 x1$ and since pF(x1, A0) is monotone decreasing in A0. It is assumed that the population has a constant size of $N \in N$ individuals and the generations change at discrete times and do not overlap. In order to compute the mean values and to identify the states that yield significant contributions, we have established a so-called tilted large deviations principle. If all moments determine the distribution. • Neighboring atoms interact. It follows that $A = x \in A \cap Qn$ Br(x) (x) is a countable union of sets from E4 and is hence in σ (E4)., Sn } - Mn $\circ \tau$. For example, characteristic functions work well with sums of independent random variables. 2.2 Independent random variables. 2.4 [fn : n \in N} \subset L1 (μ). Thus we get inductively E k . \bullet We formulate the method used in the foregoing examples as a theorem. 17.2 Discrete Markov Chains: Examples.. Thus, by Corollary 1.82, the map i is measurable with 308 14 Probability Measures on Product Spaces respect to A2 - (A1 \otimes A2). Definition 24.8 We say that a random measure X on E has independent increments if, for any choice of finitely many pairwise disjoint measurable sets A1, . , fn \in Cc+ (E) (24.1) or (IA1, . Hence, for N \geq 2c (k + 1) γ , we have w \in AN,n,i, where AN,n,i := k-1 w: w(i+l+1)/n - w(i+l)/n \leq N n-y., X(d) of X are independent random walks on Z with (i) transition probabilities P0 [X1 = xi] = 1/3 for xi = -1, 0, 1. Letting x = f - g, we get 0 =)f - g, f - g*; hence f = g. Klenke, Probability Theory, Universitext, 113 114 5 Moments and Laws of Large Numbers $\sqrt{}$ is the variance of X. Thus $E \subset DB$ for any $B \in \delta(E)$, and hence (1.3) follows. Then $|\phi(t) - \phi(s)|_2 \le 21 - \operatorname{Re}(\phi(t-s))$ for all $s, t \in \operatorname{Rd}$. Then the family (PXi, $i \in I$) of distributions of Xi is weakly relatively sequentially compact, $(\mu N) N \in N$ has a weakly convergent subsequence $(\mu Nk) k \in N$ whose weak limit will be denoted by $\mu \in M \leq 1$ (E). That is, two-dimensional symmetric simple random walk is recurrent. + Xn \leq T and compute E[N]. Hence a finite measure μ on $(\Omega, 2\Omega)$ is uniquely determined by the values $\mu(En)$, $n \in Z$. \blacklozenge The construction in the preceding example does not depend on the details of the normal distribution but only on the validity of the convolution equation N0,s+t = N0,s * N0,t . Hence (18.1) holds. if I is open to the right, if I is closed to the right, = 0. Example 9.20 Let I = N0 (or let I $\subset [0, \infty)$ be right-discrete; compare Example 9.17) and let X be an adapted real-valued stochastic process., xn-1) = $2n-11{x1 = x2 = ... = xn-1 = 0}$.) Hence H is predictable. Indeed, as in the 2.1 Independence of Events 55 preceding example, there are sets A^{*} 1, A^{*} 2, A^{*} 3 \subset {1, . Hence by the 3 dominated convergence theorem (Corollary 6.26), we have μ (C) = lime \rightarrow 0 ρ C, ε d μ i. In fact, if W is unknown, observing X gives an increasing amount of information on the true realization of W. Clearly, this construction makes use of the specific structure of the problem., fn \in Cc+ (E), A \in B([0, ∞)n) is a π -system and by Theorem 24.2 it generates M., Xk) and let An (ϕ) := 1 \in S(n) $\phi(X)$. Then P[Xn+1 = 1 X1 = x1, ..., N - 1, $\lambda xk = (1 - r)\rho k+1$ ($\theta k - \theta k$) ($\theta + \theta$)) * = (1 - r)\rho k+1 ($\theta k + 1 - \theta$ k+1) + $\theta \theta (\theta k-1 - \theta k-1) = r \rho k-1$ ($\theta k-1 - \theta k-1$) + (1 - r) $\rho k+1$ ($\theta k+1 - \theta k+1$) = r xk-1 + (1 - r) xk+1. 7.2 Inequalities and the Fischer-Riesz Theorem 167 (iv) The maps $x \rightarrow D - \phi(x)$ are monotone increasing. Assume $\mu 1$, $\mu 2 \in M(E)$ are measures with f d $\mu 2$ for all f \in Lip1 (E; [0, 1]). Example 19.18 (i) Let E = {0, 1, 2} with C(0, 2) = 0, and $A0 = \{x0\} = \{0\}$, $A1 = \{x1\} = \{2\}$. Hint: First compute the distribution of $-2 \log(U)$ and then use the transformation formula (Theorem 1.101) as well as polar coordinates. be independent real random variables in L2 (P). Then $f \le f \le f$ and hence $f d\mu \le f d\mu$. Theorem 5.28 (Kolmogorov's inequality) Let $n \in N$ and let X1, X2, . * Exercise 20.6.4 Consider a Markov chain on E = {1, 2, 3} with transition matrix p. 4.1). We prepare for the proof of Lindeberg's theorem with a couple of lemmas. Takeaways In order to check weak convergence of a sequence of probability measures, it is enough to show tightness and pointwise convergence of the characteristic functions. Definition 19.5 The system of equations (p - I)f(x) = 0, f(x) = q(x), for $x \in A$, (19.4) is called the Dirichlet problem on $E \setminus A$ with respect to p - I and with boundary value q on A., N - 1. (iv) The family $(U[-n,n]) \cap E \setminus A$ with respect to p - I and with boundary value q on A. adding superconductors to Z2. Hence sup $L(\phi)$ is convex. In particular, for i.i.d. random variables, the exchangeable σ -algebra is P-trivial., pm). The number of summands of the type $E[X|21 \cdots X|2k]$ (for different 11, . We start with a more general treatment of classes of test functions that are suitable to characterize weak convergence and then study Fourier transforms in greater detail. Lemma 1.47 Let $A \subset 2\Omega$ be an arbitrary class of sets with $\emptyset \in A$ and let μ be a nonnegative set function on A with $\mu(\emptyset) = 0$., $Xn = xn + 1 \in S$ (uk $\in S \iff xk = 1$) for every $k \le n$) r = P T1s + 1. For every $\varepsilon > 0$ and $T < \infty$, there exists a number $K < \infty$ that depends only on ε , T, α , β , C, y such that ' (P | X^{*} t - X^{*} s | $\leq K$ | t - s| y, s, t $\in [0, T] \geq 1 - \varepsilon$. n $\rightarrow \infty$ Show that n-1/2 Sn \Rightarrow N0,1 but that (Xi)i \in N does not satisfy the Lindeberg condition. In practice, all spaces that are of importance in probability theory are Polish spaces. It is a matter of taste as to which solution is preferable. Define $Am \in \{0, 1\}E$ by $\{Y \in Am\} = \{N = m\}$. 6.2 Uniform Integrability. Hint: Let E = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact
that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R) = Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R; R) is not separable. To stress this notion of a connection, we use a different symbol from the set brackets. - n n (-2 + R) = R and use the fact that Cb (R; R) is not separable. To stress this not observe a connection, w variable and let ϕ be its characteristic function. That is, it should hold that (Xti - Xti-1)i=1,...,n is independent for all 0 =: t0 < t1 < . By Theorem 1.81, Xi-1 (Ei) is a π -system that generates the σ -algebra Xi-1 (Ai) = σ (Xi). (iii) In the case $\alpha \in (0, 2)$, we have: PX is in the domain of attraction of some distribution if and only if (16.32) holds and the limit $P[X \ge x] x \rightarrow \infty P[|X| \ge x] p := \lim \text{ exists}$. Then X is called a random walk on E with weights C. Fig. Therefore, lim sup $n \rightarrow \infty$ f dµn \le lim sup $n \rightarrow \infty$ f dµn \ge lim sup h = \infty f dµn \ge lim sup h = \infty f dµn \ge lim sup h = problem that has to be resolved is to show that F (X) is a random variable. By Theorem 1.15, σ (E) is a σ -algebra. π 532 21 Brownian Motion Takeaways Brownian Motion B called integrable and we call E[X] := X dP the expectation or mean of X. (21.30) Proof "= " By Prohorov's theorem (Theorem 8.14(iv), since σ (Sm) \subset Fm, we have *) E[Sn | Sm] = E[Sn | Sm] = Sm, M > 0 (23.20) $x \in E$ Let $\delta > 0$. Thus G(0, 1) $C \in E$ (Sm) C = 1 (Sm $0 = \infty$, which shows that X is recurrent. By assumption, there is an $n \in N$ with $|Ltn(f)| < \infty$ and $|Unt(f)| < \infty$. Proof Let A1, A2, . be i.i.d. real random variables with distribution functions. Then there exists a unique σ -finite measure μ on Rn, B(Rn) such that $\mu((a, b)) = n \mu i((ai, bi))$ for all a, b \in Rn with a < b. A stochastic process X = (Xn, n \in I) is called predictable (or previsible) with respect to the filtration F = (Fn, n \in N0) if X0 is constant (if I = N0) and if for every n \in N, Xn is Fn-1 -measurable., ωn]) = n p ωi ., xn \in K with K \subset V := ni=1 B εxi (xi). (iii) Use the uniqueness theorem for Laplace transforms. l l=1 l=N+1 By Theorem 6.12(ii), (fnkl) l \in N converges to f almost everywhere on AN. More specifically, we used compactness arguments. Let Xn := 1, if the nth ball is black, 0, else. Section 4.2 (ii) A2 $\rightarrow \kappa(\omega 1, A2)$ is a (σ -)finite measure on ($\Omega 2, A2$) for any $\omega 1 \in \Omega 1$. The resulting triple (Ω , A, U Ω) is called a Laplace space. , Yn describe precisely the random variables on (Ω , A, P) from the beginning of this chapter. Hence there is a C < ∞ with μ pen – UE T V ≤ C yen for all $n \in N$, $\mu \in M1$ (E), and the best speed of convergence (in this class of transition matrices) can be obtained by choosing $\varepsilon =$ $\epsilon 0$. This statement is a special case of (ii) since fi \circ Xi is σ (Xi) – Ai -measurable (see Theorem 1.80). \bullet Theorem 2.16 (Independent generators) For any i \in I, let Ei \subset Ai be a π -system that generators) For any i \in I, let Ei \subset Ai be a π -system that generates Ai. In the first step, we determine the (random) number of jumps in (0, 1]. n This shows the lower bound (LDP 1). \in M \leq 1 (E1) with $\mu(U\phi) = 0$ and μ $- \rightarrow \mu$ weakly, then $n \rightarrow \infty \mu n \circ \phi - 1 \rightarrow \mu \circ \phi - 1$ weakly. Finally, consider the stochastic kernel κ from $\Omega 1$ to $\Omega 2$, defined by $\kappa(\omega 1, A 2) = P\omega 1$ (A2). Now we show the additional statement. 21.2 Construction and Path Property |ws - wt| $\leq c |s - t| \gamma$ for

every $s \in [0, 1]$ with $|s - t| < \delta$. We consider also as a map $N \rightarrow N$ by defining (k) = k for k > n. & Exercise 1.5.2 Give an example of two normally distributed. Note, however, that in (17.11) we have required neither independence of the random variables (Rn (x), x ∈ E) nor that all Rn had the same distribution. By the upcrossing [a, b] between times -n and 0. Theorem 21.42 (Kolmogorov's criterion for weak relative compactness) Let (Xi, i ∈ I) be a sequence of continuous stochastic processes. ∈ G and A := A1 ∪ A2 ∪ . However, we now want to develop a systematic framework for the description and construction of multi-stage experiments. 587 588 594 598 603 24 The Poisson Point Process. \bullet Example 20.28 Let $\Omega = [0, 1)$, A = B([0, 1)) and let $P = \lambda$ be the Lebesgue measure on ([0, 1), B([0, 1))). Show that A is a continuous linear operator from L2 (μ 1) to L2 (μ 2). We show the optional stopping theorems in the second section. (15.4) k=1 Hint: (i) Define the characteristic functions (see Theorem 15.13) ϕ_1 (t) = ϕ_2 (t) = (1 - t/2) + . \bullet Theorem 13.23 Let μ , μ_1 , μ_2 , . (ii) E is complete and there is a summable sequence (ϵ_n) $h \in A$ for all $n \in N$. Let A, A1, A2, . (i) Show the validity of Helly's theorem with V replaced by Vd . Define X := AW + µ. An F-martingale Y is thus determined uniquely by the terminal values YT (and vice versa). Each path of the process X of partial sums that ends above a corresponds to a unique path that reaches a but ends below a. Hence, what is the fair price $\pi(VT)$ for which a trader would offer (and buy) the contingent claim VT? Then { $|f| > \alpha \epsilon$ } $|f| | d\mu < {|f| > \alpha \epsilon}$ $|f| | d\mu + { g\epsilon/2 > \alpha \epsilon} g\epsilon/2 d\mu < \epsilon$. (ii) X is mixing if and only if X is aperiodic. Our first task is to give precise definitions. For general measurable f, the statement follows by the usual approximation arguments. Essentially it is necessary and sufficient that the state space of the chain cannot be decomposed into subspaces • that the chain does not leave • or that are visited by the chain periodically; e.g., only for odd n or only for even n. To this end, we check (i)-(iii) of Definition 1.2. (i) Clearly, $\Omega \in AI$. Let X be binomially distributed, X ~ bn,p. n $\rightarrow \infty$ Remark 13.32 The implication (ii) in Theorem 13.29 is less useful but a lot simpler to prove. i=1 A function $f: I \rightarrow R$ is called Riemann integrable if there exists a t such that the limits of the lower sums and upper sums are finite and coincide. For $n \in N0$, define Xn := nm=1 Ym . + Tn-1 and $Xt = sup\{n \in N0 : Sn \le t\}$. Definition 1.1 A class of sets A is called $\cdot \cap$ -closed (closed under sums are finite and coincide. For $n \in N0$, define Xn := nm=1 Ym . + Tn-1 and $Xt = sup\{n \in N0 : Sn \le t\}$. intersections) or a π -system if $A \cap B \in A$ whenever $A, B \in A, \infty \bullet \sigma$ - \cap -closed (closed under countable) intersections) if n=1 An $\in A$ for any choice of countably many sets A1, A2, . The function p in (v). Proof Clearly, every dn is a complete metric on (C([0, n]), ∞). \bullet Remark 12.9 Denote by $T = n \in N \sigma$ (Xn+1, Xn+2, The case $\mu = \delta 0$ is trivial. We show that for any $f \in C b$ (E; R), any $x \in E$ and any $\epsilon > 0$, there exists a $gx \in C$ with gx(x) = f(x) and $gx(y) \leq f(y) + \epsilon$ for all $y \in E$. Theorem 2.47 (Uniqueness of the infinite open cluster) For any $p \in [0, 1]$, we have $Pp[N \leq 1] = 1$. In order to apply the Poisson approximation theorem (Theorem 3.7), for fixed $n \in N$, we decompose the intervals of equal length, * I n (k) := (k - 1)2-n t, k2-n t, k = 1, . Intuitively, this should be x n. As the minimum of x independent exp1 -distributed. We say that a point $x \in Zd$ is a trifurcation point if • x is in an infinite open cluster C p(x), • there are exactly three open edges with endpoint x, and • removing all of these three edges splits C p(x) into three mutually disjoint infinite open clusters. $n \to \infty$ "(iv) \Rightarrow (iii)" (for finite μ) Assume that $\mu(A) < \infty$ for every $A \in A$ and that μ is \emptyset -continuous. Furthermore, the text has been extended carefully in many places. & Exercise 21.1.4 Let $X = (Xt) t \ge 0$ be a stochastic process on (Ω, F, P) with values in the Polish space E and with right continuous paths. Then (aX + bY) is a martingale. "(ii) \Rightarrow (iii)" Assume (ii). Show that the Chebyshev polynomials of the second kind are orthonormal with respect to ν ; that is, Um Un dv = 1{m=n}. k=0 We get $|f(Sn/n) - f(p)| \le \epsilon + 2f \propto 1{|(Sn/n) - p| \ge \delta}$ and thus (by Theorem 5.14 with V = p(1 - p) \le 14) |fn(p) - f(p)|] + Sn \le \epsilon + 2f \propto P - p \ge \delta n \le \epsilon + f \propto 2 \delta 2 n n \rightarrow \infty for any $p \in [0, 1]$. $\tau k+1$ }, we have $(H \cdot Y)j = (H \cdot Y)\sigma k$. In particular, a random walk on Z with centered increments is recurrent (Chung-Fuchs theorem, compare Theorem 17.41). Example 20.12 Let (Xn)n \in N0 be i.i.d. and let Xn (ω) = X0 (τ n (ω)). Definition 14.20 Let n \in N. Denote by LI := LC I := 1 I (x, y) 2 R(x, y) 2 x, y \in E the energy dissipation of I in the network (E, C). The ith individual in the nth generation has Xn, i offspring (in the (n + 1)th generation). Also one A be a semiring. However, it is not a Markov chain with respect to the so-called annealed measure $P[X \in \cdot]$, m and distribution p. (i) The distributic p. (i) The distributi reason why we studied independence of classes of events in the last section. \blacklozenge Example 15.19 Let $\phi(t) = (1 - 2|t|/\pi) + be$ the characteristic function from the preceding example. Theorem 5.17 (Etemadi's strong law of large numbers (1981)) Let X1, X2, Show that X is a random measure if and only if we have $P[X(B) 0, \text{there exists a sequence (H n)} \in \mathbb{S}$ (respectively mX ($-\varepsilon 0$) < ∞). The event B from above can be written as B = Zn+1 < Zn only finitely often. \blacklozenge Example 1.63 (Infinite product measure, continuation of Example 1.40) Let E be a finite set and let $\Omega = E$ N be the space of E-valued sequences. We have extended this notion to families of events and even to families of sets. Then f is harmonic on E \ A. If (Xi)i \in I is independent, then (fi \circ Xi)i \in I is independent. Proof Clearly, E[X] is finitely additive. Exercise 21.5.4 Let $d \in N$. (1) Choose a random coordinate I according to some distribution (qi) i A. Generating functions are the ideal tool for the analysis of such processes. Assume that $E[(Gn (t) - Gn (s))4] \leq C (t - s)2 + |t - s|/n$ for some C > 0. Define $E := \infty n = 1$ Kn. In general, the growth to infinity of Zn can also be slower than mn. In this section, we develop a formal framework for the quantification of probabilities of rare events in which the complete theory of large deviations can be developed. (iii) First assume $X \ge 0$ and $Y \ge 0$. Assume that there is an integrable dominating function $0 \le g \in L1$ (μ) $n \rightarrow \infty$ with $|fn| \le g$ almost everywhere for all $n \in N$. Definition 21.2 Let (E, d) and (E, d) be metric spaces and $\gamma \in (0, 1]$. = 1, let dn = n an E[sin(X/an)] for all $n \in N$. In Sect. F \in Fn Then F := ∞ n=1 F \in Fn ∞ $\mu(F) \leq n=1$ F (n=1) F = max lim sup log Pn ($-\infty, x-1$), lim sup $n \rightarrow \infty$ n = max - I (x-), -I (x+) = - inf I (C). (ii) This is obvious since P $\phi(X)$ = PX $\circ \phi - 1$. Hence X is ergodic. Corollary 6.13 Let (E, d) be a separable metric space. For $\omega 0 \in \Omega 0$ and $n \in N$, let $A \omega 0$, $n := \{\omega 1 \in \Omega 1 : \kappa^2 ((\omega 0, \omega 1), \Omega 2) < n\}$. First we consider the situation of a simple shift: Let $\Omega = E N 0$, where E is a finite set equipped with the product σ -algebra A = (2E) $\otimes N0$. By the monotone convergence theorem (Theorem 4.20), we infer E[|XY|] = lim E[XN] N \rightarrow \infty lim E[XN] N $\rightarrow \infty$ lim E[XN] N $\rightarrow \infty$ lim E[XN] N $\rightarrow \infty$ lim E[XN] = lim E[XN] N $\rightarrow \infty$ lim E[XN] N \rightarrow \infty lim E[XN] N $\rightarrow \infty$ lim E[XN] N \rightarrow \infty lim E[XN] N \rightarrow \infty lim E[XN] N \rightarrow \infty lim E (Solomon [158]) Assume that $E[|\log(0)|] < \infty$. For any measurable set $A \subseteq E$, define $\mu(A) := \sup u^n n+1$ ($A \cap Wn$). For a formal proof along the lines of this heuristic, see Sect. Clearly, P[B|A] = E[1B|A] for all $B \in A$. Warning: One of the implications is rather difficult to show. 19.6 whose effective resistance Reff 3 Z. At each step, choose one of the N balls uniformly at random. Hence also U := V \cap W is open. Analogously, define P[A|X = x] = E[1A X = x] for A \in A. and n 2 n tt 2 1 - t $-\phi \sqrt{-\phi} \sqrt{-\phi$ Lebesgue integral of f. Proof Let $\Omega = \Omega + \Omega - be$ a Hahn decomposition. The dashed vertical line indicates the critical inverse temperature. (21.16) It is enough to consider continuous bounded functions F that depend on only finitely many coordinates t1, By we obtain n lim E[$e\lambda Zn/2$] = $n \rightarrow \infty 1$ 1 1 1 + $\lambda = + = E[e - \lambda W]$, 1 + 2 λ 2 2 1 + 2 λ where in the last step we assumed that $PW = 12 \delta 0 + 12 \exp[/2]$. This implies (using the triangle inequality; see Theorem 5.3(v)) $\infty \infty$)***))*) E |ST| = E |Sn| 1{T = n} = E |Sn| 1{T 1, we get *k *k) $P[B \in AN, n, i] = P|B1/n| \le N n - \gamma = P|B1| \le N n - \gamma + 1/2 \le N k nk(-\gamma + 1/2)$, $n, j * F\{i\}$ (x) = $P[Xi \le x] = P[Xi \le x] = P[Xi$ $\sim N\mu, C \iff (iii) \iff (iv)$. \blacklozenge Takeaways Consider a two-step random experiment where in the first step we choose a probability measure $\Xi \infty$ on some space E. $n \rightarrow \infty$ (iv) Finally, show that $Mn \Rightarrow M$. Step 7. \in A with $A \subset \infty i=1$ Ai. The same is true for (iii) and (iv). Definition 13.12 (Weak and vague convergence) Let E be a metric space. , n since u is even. (Of course, this is wishful thinking. (19.6) Equation (19.6) is sometimes called the equation of detailed balance. In practice, for a large space E, computing the spectral gap is often extremely difficult. For $x \in \mathbb{R}d$ and $A \subset \mathbb{R}d$, define Ex := m-1 ($\{x\}\} = \{v \in E : m(v) = x\}$ and EA = m-1 ($A = \{v \in E : m(v) \in A\}$.

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